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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

National Certificate of Educational Achievement
TAUMATA MĀTAURANGA Ā-MOTU KUA TAEA

Level 3 Calculus, 2006

90635 Differentiate functions and use derivatives to solve problems

Credits: Six
9.30 am Wednesday 29 November 2006

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Make sure you have a copy of the Formulae and Tables booklet L3-CALCF.

You should answer ALL the questions in this booklet.

Show ALL working for ALL questions.

Show any derivatives that you need to find when solving the problems.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

For Assessor's use only	Achievement Criteria	
Achievement	**Achievement with Merit**	**Achievement with Excellence**
Differentiate functions and use derivatives to solve problems. ✓	Demonstrate knowledge of advanced concepts and techniques of differentiation and solve differentiation problems. ✓	Solve more complex differentiation problem(s). —
Overall Level of Performance		M

This Student is merit because they achieved 2×A1, 2×A2, 3×M, using a mainly right E question as further evidence for the 3rd M.

A Not excellence. Too many wrong answers Q 6, 7, 8

B To get excellence needed to be more proficient in solving differentiation problems

C. At first glance this looked like a weak merit but in reality, he has missed both M2s for reasons that have nothing to do with his understanding of differentiation. Understanding of problems + choice of methods to solve them were spot on.

You are advised to spend 50 minutes answering the questions in this booklet.

Assessor's use only

QUESTION ONE

Differentiate the following functions.
You do not need to simplify your answers.

(a) $y=(x^2-3x)^5$

$\frac{dy}{dx} = 5(x^2-3x)^4(2x-3)$

N

(b) $y=5\cot 2x$

$\frac{dy}{dx} = -10\operatorname{cosec}^2(2x)$

A1

(c) $y=\dfrac{\sin x}{x+3}$

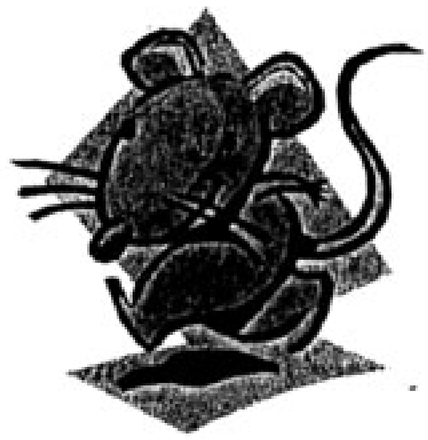
$\frac{dy}{dx} = \frac{\cos x(x+3)-\sin x}{(x+3)^2}$

A1

1a) Sloppy. Left off the power of bracket. Would have forgotten & not picked it up when checking.

1b) 1c) } as expected.

QUESTION TWO



A teenager breeds mice for pet shops.
The number of mice for the first nine months of his production can be modelled by:

$$N(t) = 10e^{0.5t} + 12\ln(2t+7),\ 0 \le t \le 9$$

where N is the total number of mice
and t is the time in months.

At what rate is the number of mice increasing at 7 months?

Show any derivatives that you need to find when solving this problem.

$\frac{dN}{dt} = 10 \times 0.5e^{0.5t} + 12 \times \frac{2}{2t+7}$ $\quad t = 7$

$10 \times 0.5 \times e^{0.5 \times 7} + \frac{24}{2 \times 7 + 7}$

$= 5e^{3.5} + \frac{24}{21}$

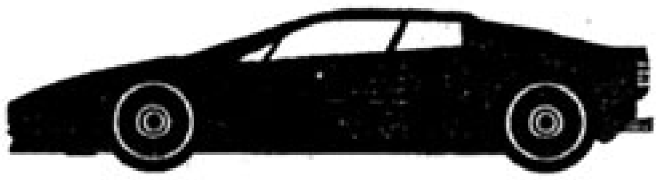
$= 166.7$

$\therefore$ mice increase at rate of 166.7 at 7 months.

Both Q2, Q3

Correct derivatives
used appropriately
to solve problems.

QUESTION THREE



The power of an engine of a sports car can be modelled by the function:

$$P(x) = 480 - \frac{500\,000}{x} - \frac{x}{25}, \; 1200 \le x \le 6000$$

where P is the power of the engine (in kilowatts)
and x is the speed of the engine (in revolutions per minute).

Calculate the speed of the engine that generates the maximum power.

You may assume $\frac{d^2P}{dx^2} < 0$.

Show any derivatives that you need to find when solving this problem.

$P(x) = 480 - 500\,000x^{-1} - \frac{1}{25}x$

$P'(x) = 500000x^{-2} - \frac{1}{25} = 0$

$\frac{500\,000}{x^2} = \frac{1}{25}$, $\quad x^2 = 500000 \times 25$

$x = \pm 3535.5$ (5sf)

$P''(x) = -1000\,000\,x^{-3} < 0$

$\frac{-1000\,000}{(3535.5)^3} < 0$ ✓

∴ 3535.5 revolutions / mins.

Assessor's use only

A2

QUESTION FOUR

Use the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to show that the derivative of $f(x) = 3x^2 + x + 5$ is $f'(x) = 6x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 + (x+h) + 5 - (3x^2 + x + 5)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6x\cancel{h} + 3h^{\cancel{2}} + \cancel{x} + \cancel{h} + \cancel{5} - \cancel{3x^2} - \cancel{x} - \cancel{5}}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + 3h + 1$$

$$= 6x + 3 \times 0 + 1$$

$$= 6x + 1$$

$$\therefore f'(x) = 6x + 1$$

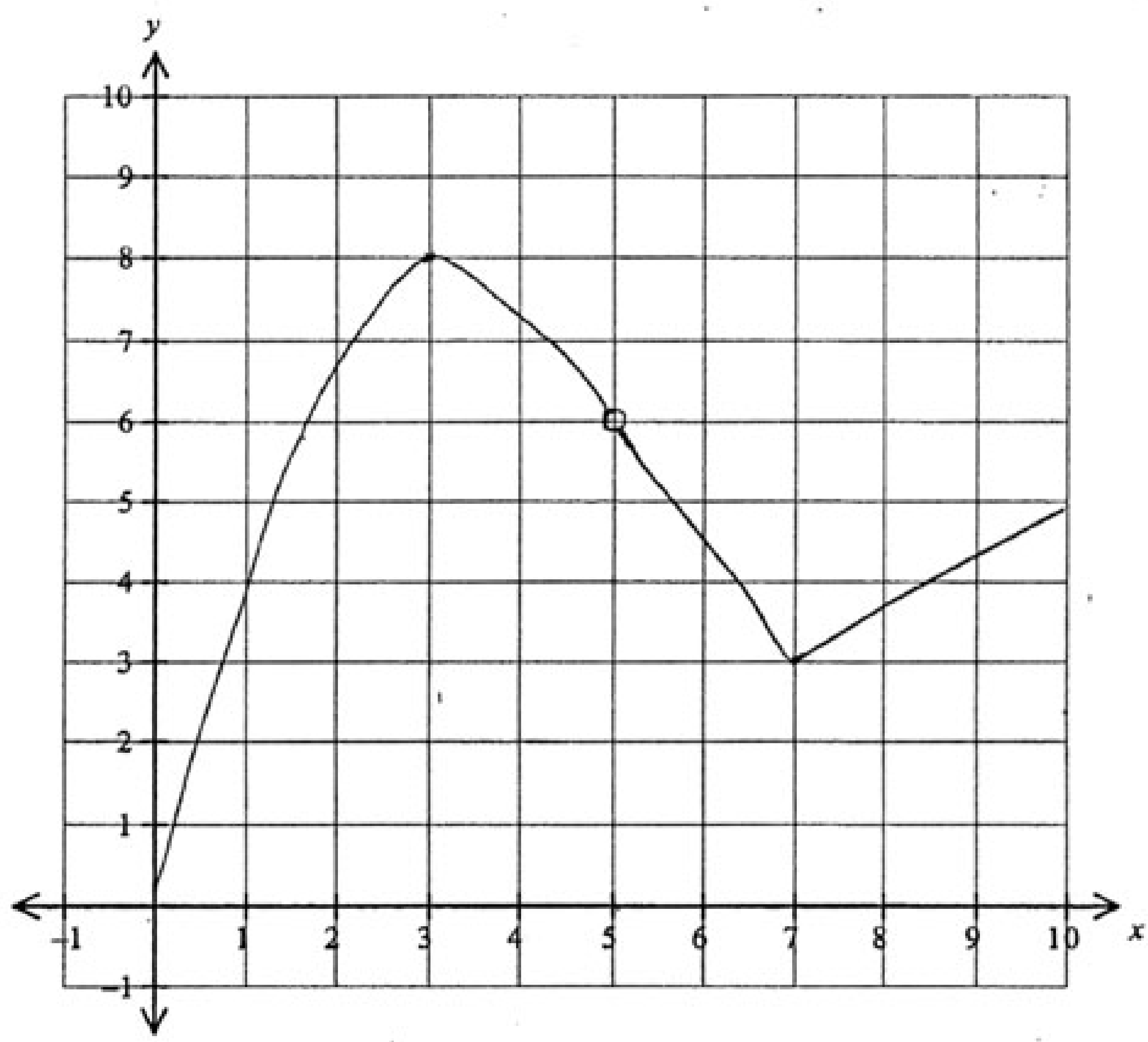
Both Q4 and Q5 exactly as assessment schedule described the evidence.

QUESTION FIVE

On the axis below, sketch a graph of $f(x)$ that

- is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$
- is concave down ($f''(x) < 0$) for $0 < x < 5$
- has $f'(x) = 0$ at $(3,8)$ (max) or ~~min or inf~~
- has $\lim_{x \to 5} f(x) = 6$
- is not differentiable at $(7,3)$.

If you need to redraw this graph, use page 10.

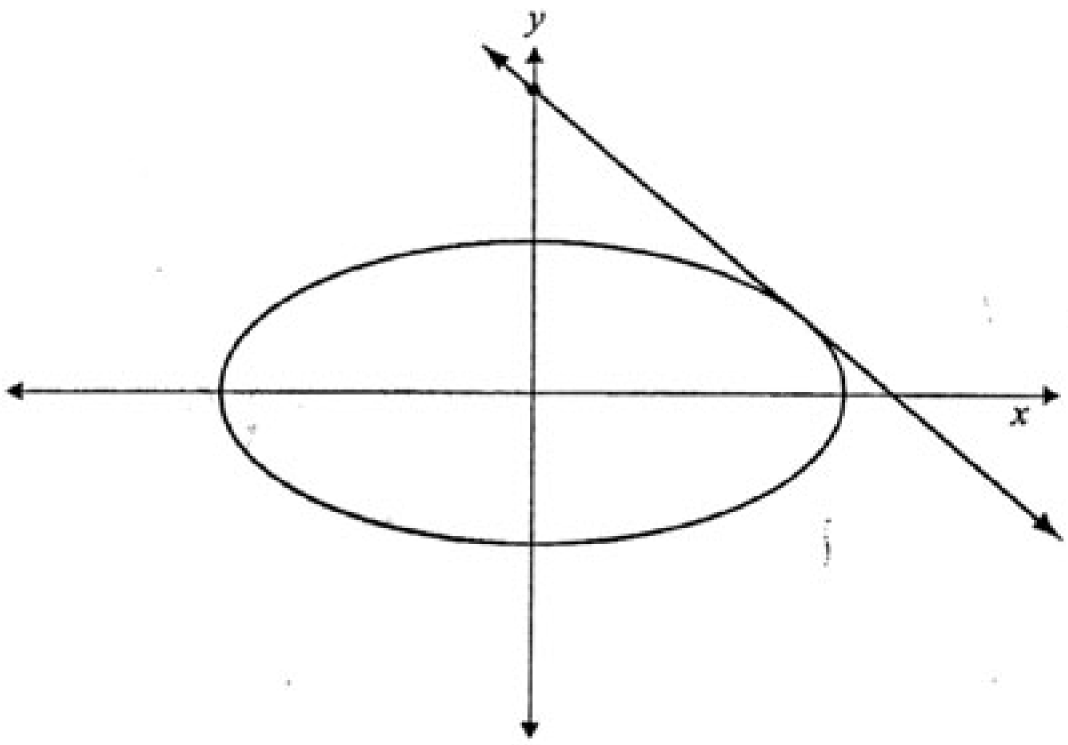


Assessor's use only

QUESTION SIX

The graph below shows the curve defined by the parameters $x = 6\cos t$ and $y = 4\sin t$.

It also shows the tangent to the curve at the point $t = \frac{\pi}{6}$.



Find the y-intercept of the tangent to the curve when $t = \frac{\pi}{6}$.

Show any derivatives that you need to find when solving this problem.

$\frac{dx}{dt} = -6\sin t \qquad \frac{dy}{dt} = 4\cos t \qquad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{4\cos t}{-6\sin t}$

$x = 6\cos t$

$= 6\cos\frac{\pi}{6} = 5.9997 \qquad = \frac{4\cos\frac{\pi}{6}}{-6\sin\frac{\pi}{6}}$

$y = 4\sin t$

$= 0.03655 \qquad = -72.95$

$y - 0.03655 = -72.95(x - 5.9997)$

when $x=0 \quad y = -72.95x - 5.9997 + 0.03655$

$y = 437.71$

$\therefore (0, 437.71)$

$\frac{dy}{dx}$ is worked out correctly to the line

$$\frac{4\cos\frac{\pi}{6}}{-6\sin\frac{\pi}{6}}$$

but calculation is wrong Should be -1.15 not -72.95 This has come from leaving his calculator in degrees mode when clearly it should be in radians.

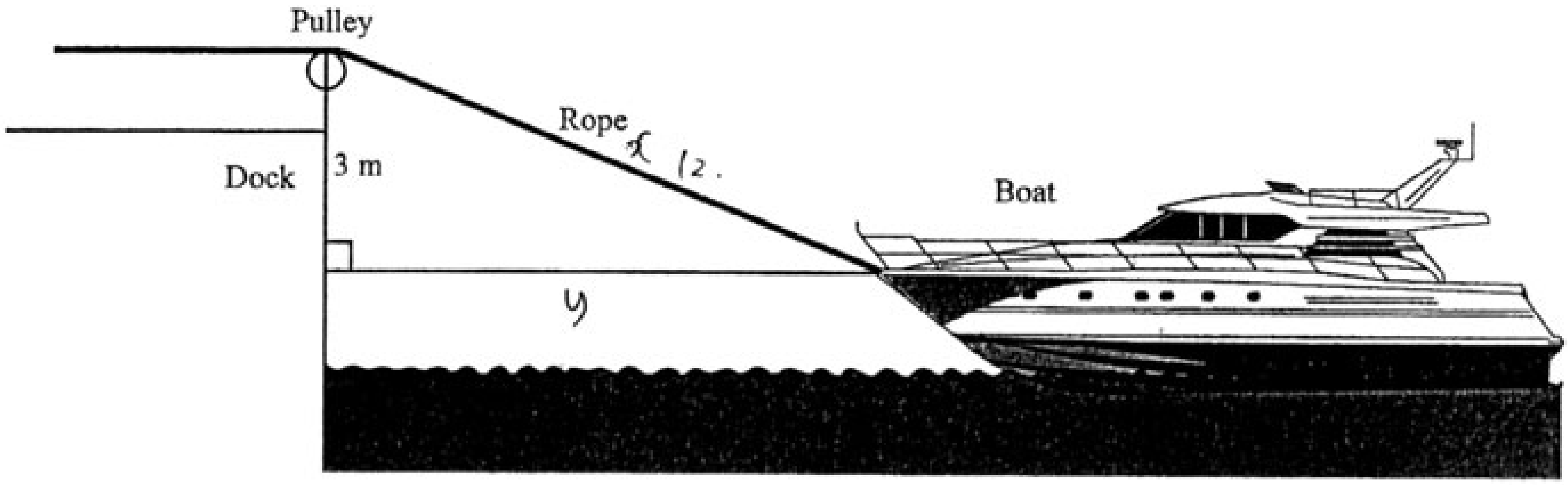
The method used by the student is correct and well set out.

However a merit student should have wondered at a final answer of 437.71 and checked the angle mode of his calculator.

Assessor's use only

QUESTION SEVEN

A boat is pulled into dock by means of a rope running through a pulley on the dock.
The rope is attached to the bow of the boat at a point 3 metres below the level of the pulley.
The rope is being pulled through the pulley at a rate of 8 metres per minute.



At what rate will the boat be approaching the dock when there is 12 metres of rope between the boat and the pulley?

Show any derivatives that you need to find when solving this problem.

$\frac{dx}{dt} = 8$ m/m $\qquad \frac{dy}{dt} = ? \qquad x = 12$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{1}{\sqrt{x^2-9}} \times 8$

$y^2 = x^2 - 3^2$

$y = \sqrt{x^2-9}$

$= (x^2-9)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x^2-9)^{-\frac{1}{2}} \times 2x$

$= \frac{1}{\sqrt{x^2-9}}$

$x = 12$

$\frac{dy}{dt} = \frac{8}{\sqrt{12^2-9}} = 0.6885$ (4sf)

∴ the ~~boat is~~ approaching the rate of 0.6885 m/s when there is 12 m of rope.

N2

Again the student has made an error that could have been picked up with good checking

$$\frac{dy}{dx} = \frac{1}{2}(x^2-9)^{-\frac{1}{2}} \cdot 2x$$

He has lost the x

So $\frac{dy}{dt}$ is $\frac{8}{\sqrt{x^2-9}}$ instead of $\frac{8x}{\sqrt{x^2-9}}$

He has substituted correctly for x into the wrong $\frac{dy}{dt}$.

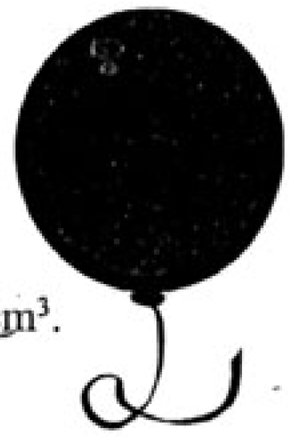
Student understands problem. Knows what he has to do to solve it.

Setting out is clear & logical

There is an error in $\frac{dy}{dx}$ and hence $\frac{dy}{dt}$ which is <u>not</u> minor.

QUESTION EIGHT

Assessor's use only



A spherical balloon is being filled with air at a constant rate.
At a certain time, the volume is 360 cm³.
20 seconds later, the volume is 450 cm³.

Find the rate of increase of the surface area when the balloon has a volume of 1500 cm³.

Show any derivatives that you need to find when solving this problem.

$V = \frac{4}{3}\pi r^3$ $\quad SA = 4\pi r^2$ $\quad t = 0 \quad V = 360$

$\boxed{\frac{dSA}{dt} = \frac{dSA}{dr} \times \frac{dV}{dt} \times \frac{dr}{dV}}$ $\quad$ when $V = 1500$ $\quad t = 20 \quad V = 450$

$\frac{dV}{dt} = \frac{450 - 360}{20} = 4.5$

$\frac{dV}{dr} = 4\pi r^2 \qquad \frac{dSA}{dr} = 8\pi r$

$\frac{dSA}{dt} = \cancel{8\pi r}^{2} \times 4.5 \times \frac{1}{\cancel{4\pi r^2}}$ $\qquad V = 1500 = \frac{4}{3}\pi r^3$

$r^3 = \frac{1500}{4\pi} \times 3$

$= \frac{9}{r}$ $\qquad = 3534.29 \ (2dp)$

$r = (3534.29)^{\frac{1}{3}}$

$= \frac{9}{15.23}$ $\qquad = 15.23 \text{ cm} \ (2dp)$

$= 0.591 \ (3sf)$

$\therefore \frac{dSA}{dt} = 0.591 \text{ cm}^2/\text{sec.}$

C

M2

Again good understanding of question and method needed to solve it.

The error is in the calculation of r. This student was not able to solve $\frac{4}{3}\pi r^3 = 1500$ for r. His r^3 statement is correct but has been inputted into his calculator incorrectly to arrive at 3534^{29}

$$r^3 = \frac{1500}{4\pi} \times 3$$

$$r^3 = 3534.29$$

To arrive at this number, candidate has multiplied by π, instead of dividing by π.