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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

National Certificate of Educational Achievement
TAUMATA MĀTAURANGA Ā-MOTU KUA TAEA

Level 3 Calculus, 2006

90635 Differentiate functions and use derivatives to solve problems

Credits: Six
9.30 am Wednesday 29 November 2006

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Make sure you have a copy of the Formulae and Tables booklet L3-CALCF.

You should answer ALL the questions in this booklet.

Show ALL working for ALL questions.

Show any derivatives that you need to find when solving the problems.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

For Assessor's use only	Achievement Criteria	
Achievement	**Achievement with Merit**	**Achievement with Excellence**
Differentiate functions and use derivatives to solve problems. ✓	Demonstrate knowledge of advanced concepts and techniques of differentiation and solve differentiation problems. ✓	Solve more complex differentiation problem(s). ✓
Overall Level of Performance		E

You are advised to spend 50 minutes answering the questions in this booklet.

Assessor's use only

QUESTION ONE

Differentiate the following functions.
You do not need to simplify your answers.

(a) $y=(x^2-3x)^5$

$5 \cdot (2x-3) \cdot (x^2-3x)^4$

A1

(b) $y=5\cot 2x$

$2 \cdot 5(-\operatorname{cosec}^2 2x)$

A1

(c) $y=\dfrac{\sin x}{x+3}$

$\dfrac{\cos x\,(x+3) - \sin x \cdot 1}{(x+3)^2}$

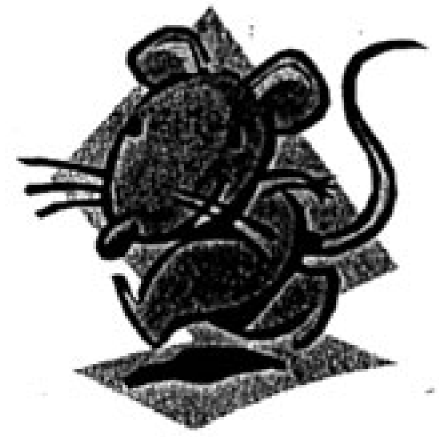
A1

All correct.
Dots for times higher on line than normal but still clear in their meaning.

Teachers usually teach dot on line for times, dot up in the middle is a decimal point.

Assessor's use only

QUESTION TWO



A teenager breeds mice for pet shops.
The number of mice for the first nine months of his production can be modelled by:

$$N(t) = 10e^{0.5t} + 12\ln(2t+7),\ 0 \le t \le 9$$

where N is the total number of mice
and t is the time in months.

At what rate is the number of mice increasing at 7 months?

Show any derivatives that you need to find when solving this problem.

$$\frac{dN}{dt} = 10 \times 0.5\, e^{0.5t} + \frac{12 \times 2}{2t+7}$$

$$= 5e^{0.5t} + \frac{24}{2t+7}$$

$\therefore\ t = 7$

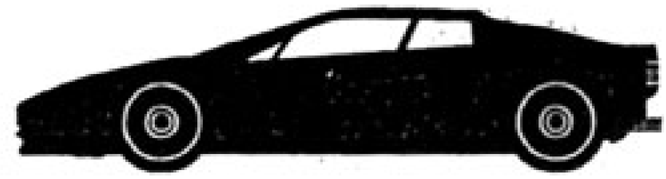
$$= 5 \times e^{0.5 \times 7} + \frac{24}{2 \times 7 + 7}$$

$$= 166.72 \approx 167.$$

$\therefore$ the rate of mice increasing at $166.72\ Nm^{-1}$ ≈ 167

Clearly defined derivatives
Excellent setting out. Spot on!

QUESTION THREE



Assessor's use only

The power of an engine of a sports car can be modelled by the function:

$$P(x) = 480 - \frac{500\,000}{x} - \frac{x}{25}\ ,\ 1200 \le x \le 6000$$

where P is the power of the engine (in kilowatts)
and x is the speed of the engine (in revolutions per minute).

Calculate the speed of the engine that generates the maximum power.

You may assume $\frac{d^2P}{dx^2} < 0$.

Show any derivatives that you need to find when solving this problem.

$P'(x) = -500000x^{-1} - \frac{1}{25}$

$= 500000\,x^{-2} - \frac{1}{25}$.

$\cancel{500000x^{-2} - \frac{1}{25} = 0}$

[illegible]

$\frac{500000}{x^2} = \frac{1}{25}$

$\therefore\ x^2 = \frac{500000}{\frac{1}{25}}$

$\therefore\ x = 3535.53$

$\because\ \frac{d^2P}{dx^2} < 0$

$\therefore\ x = 3535.53$ as the maximum point

A2

$\therefore\ P = 480 - \frac{500000}{3535.53} - \frac{3535.53}{25}$

$= 197.16$ kw

$\therefore$ the power is 197.16kw

QUESTION FOUR

Use the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to show that the derivative of $f(x) = 3x^2 + x + 5$ is $f'(x) = 6x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 + (x+h) + 5 - (3x^2 + x + 5)}{(h)}$$

$$= \lim_{h \to 0} \frac{3(x^2 + h^2 + 2hx) + x + h + 5 - 3x^2 - x - 5}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 3h^2 + 6hx + \cancel{x} + h \cancel{+5} - \cancel{3x^2} \cancel{-x} \cancel{-5}}{h}$$

$$= \lim_{h \to 0} 3h + 6x + 1$$

$$= 6x + 1$$

Someone who understands the importance of limit statements when using first principles to find $f'(x)$

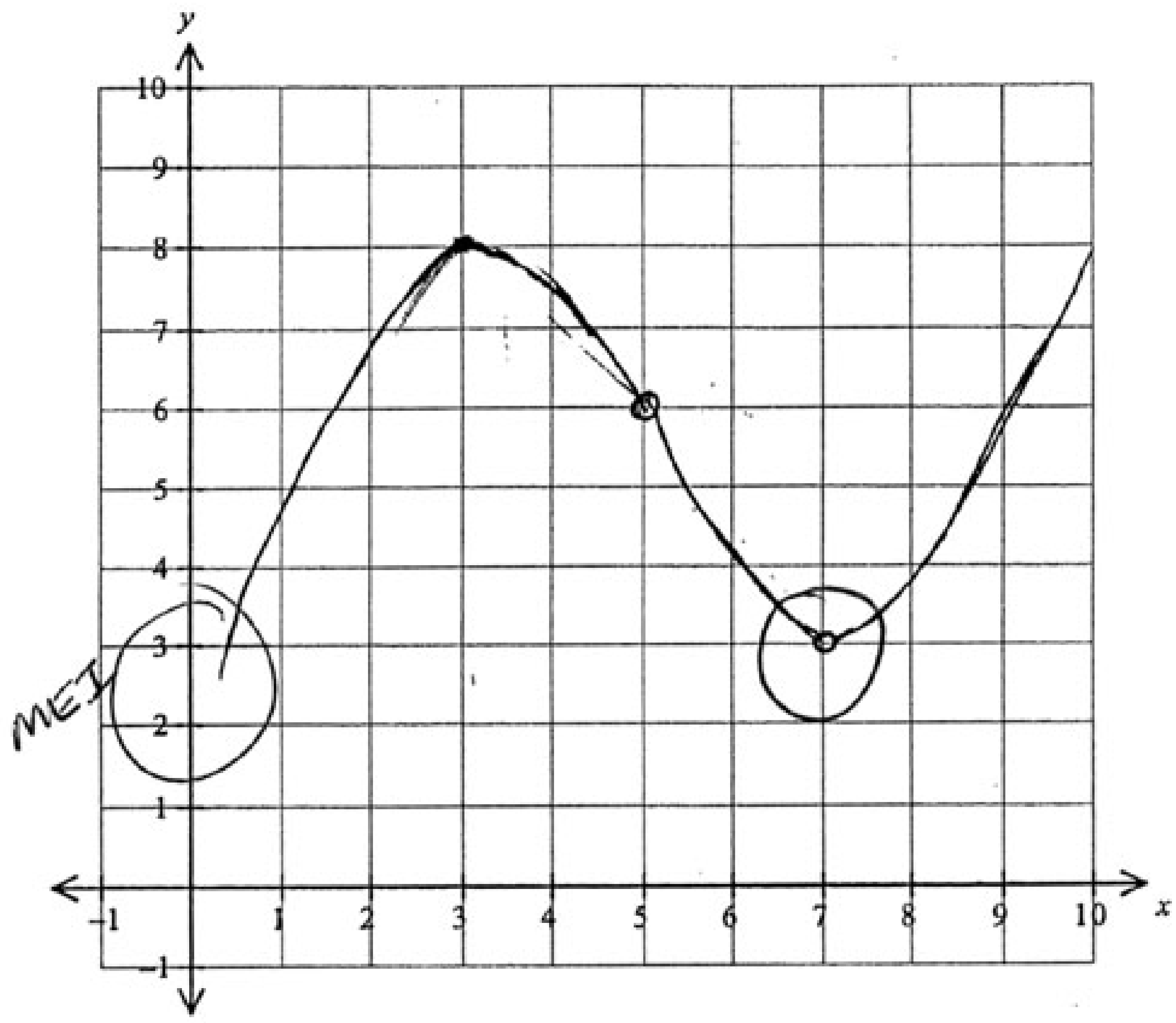
QUESTION FIVE

Assessor's use only

On the axis below, sketch a graph of $f(x)$ that

- is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$
- is concave down ($f''(x) < 0$) for $0 < x < 5$ ✓
- has $f'(x) = 0$ at (3,8) ✓
- has $\lim_{x \to 5} f(x) = 6$ ✓
- is not differentiable at (7,3). ✓

If you need to redraw this graph, use page 10.



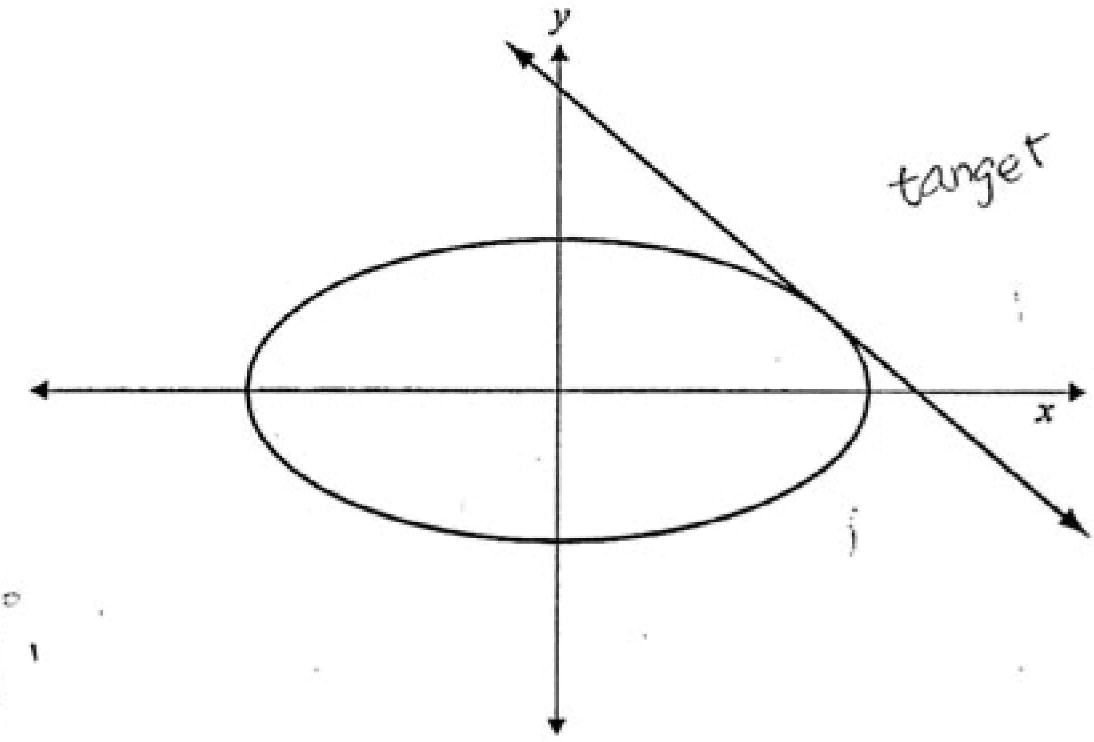
M1

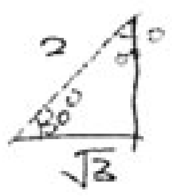
Does not understand how a function can be continuous and not differentiable at the same time.

QUESTION SIX

The graph below shows the curve defined by the parameters $x = 6\cos t$ and $y = 4\sin t$.

It also shows the tangent to the curve at the point $t = \frac{\pi}{6}$.





Find the y-intercept of the tangent to the curve when $t = \frac{\pi}{6}$.

Show any derivatives that you need to find when solving this problem.

$x = 6\cos(\frac{\pi}{6}) = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$y = 4\sin(\frac{\pi}{6}) = 4 \cdot \frac{1}{2} = 2$

$\frac{dx}{dt} = -6\sin t$

$\frac{dy}{dt} = 4\cos t \qquad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 4\cos t \times \frac{1}{(-6\sin t)} = \frac{4\cos t}{-6\sin t} = \frac{2\cos t}{-3\sin t}$

$= \frac{2 \cdot \frac{\sqrt{3}}{2}}{-3 \cdot (\frac{1}{2})} = \frac{2\sqrt{3}}{2} \Big/ (-\frac{3}{2}) = \frac{2\sqrt{3}}{2} \times (-\frac{2}{3})$

$= -\frac{2}{3}\sqrt{3}$

$~~y = 8~~ \quad (y-2) = (x - 3\sqrt{3}) \cdot (-\frac{2}{3}\sqrt{3})$

$y = \frac{-2\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3} \cdot 3\sqrt{3} + 2$

$= \frac{-2\sqrt{3}}{3}x + 6 + 2$

$y = \frac{-2\sqrt{3}}{3}x + 8.$

Nicely set out. Very capable in use of fractions and surds.

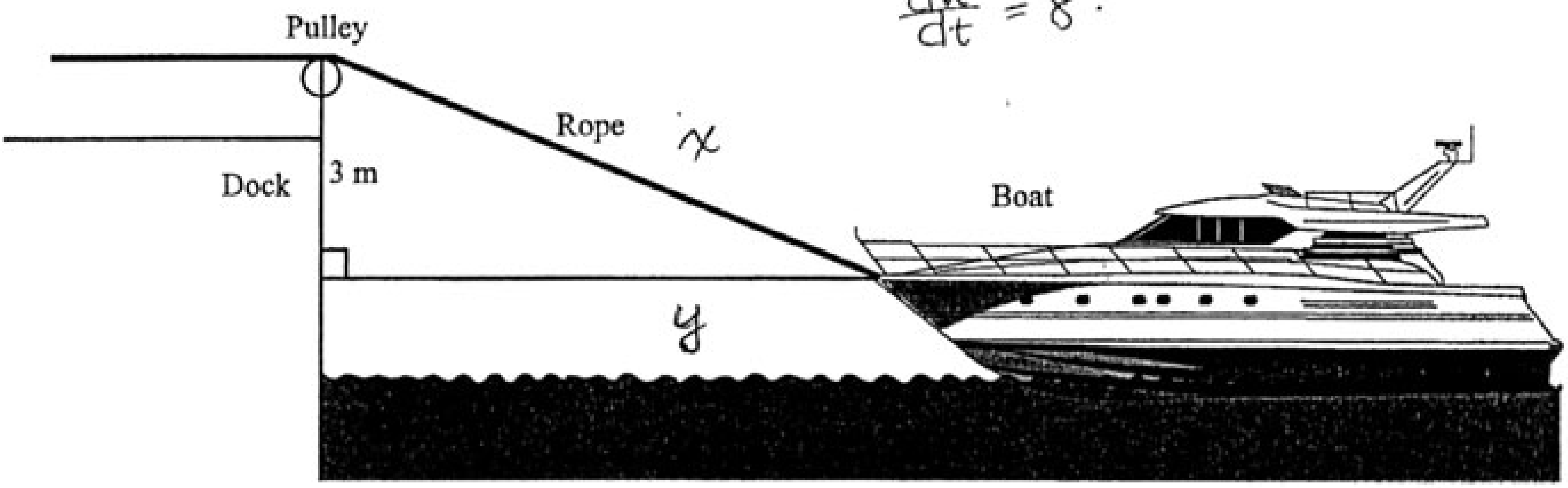
Assessor's use only

QUESTION SEVEN

Assessor's use only

A boat is pulled into dock by means of a rope running through a pulley on the dock.
The rope is attached to the bow of the boat at a point 3 metres below the level of the pulley.
The rope is being pulled through the pulley at a rate of 8 metres per minute.

$\frac{dx}{dt} = 8$.



At what rate will the boat be approaching the dock when there is 12 metres of rope between the boat and the pulley?

Show any derivatives that you need to find when solving this problem.

$\frac{dy}{dt} = ?\ \frac{dx}{dt} \cdot \frac{dy}{dx}$

$\therefore\ y^2 = x^2 - 9$.

$y = (x^2-9)^{\frac{1}{2}}$

$\frac{dy}{dx} = \cancel{2}x \cdot \frac{1}{\cancel{2}}(x^2-9)^{-\frac{1}{2}}$

$= x(x^2-9)^{-\frac{1}{2}}$

$\frac{dy}{dt} = x(x^2-9)^{-\frac{1}{2}} \cdot 8$

$= 12(144-9)^{-\frac{1}{2}} \cdot 8$

$= 8.26$.

$\therefore$ The rate of boat approaching at 8.26m

M2

Standard setting out. Exactly as schedule

$\frac{dA}{dt} = \frac{dV}{dt} \cdot \frac{dA}{dV}$ $\quad \frac{dA}{dV} = \frac{dA}{dr} \cdot \frac{dr}{dV}$

$\frac{dA}{dr} \cdot \quad \frac{dV}{dt}$

QUESTION EIGHT

A spherical balloon is being filled with air at a constant rate.
At a certain time, the volume is 360 cm^3.
20 seconds later, the volume is 450 cm^3.

$A = -$

Find the rate of increase of the surface area when the balloon has a volume of 1500 cm^3.

Show any derivatives that you need to find when solving this problem.

$\frac{dV}{dt} = \frac{450 - 360}{20} = 4.5\ cm^3 s^{-1}$

$A = 4\pi r^2 \therefore \frac{dA}{dr} \cdot 8\pi r$, $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{3} \times 3\pi r^2 = 4\pi r^2$

$\frac{dA}{dt} = \cancel{\frac{dV}{dt}} \times \cancel{\frac{dA}{dr}} \times \cancel{\frac{dr}{dV}}$ $\quad \frac{dr}{dV} = \frac{1}{4\pi r^2}$

$= 4.5 \times (8\pi r) \times \frac{1}{4\pi r^2}$

$= 4.5 \times 8 / 4r = \frac{4.5 \times 2}{r} = \frac{9}{r}$

$\because 1500 = \frac{4}{3}\pi r^3 \therefore r = 7.1$

$\therefore \frac{dA}{dt} = \frac{9}{7.1} = 1.27\ cm^2 s^{-1}$

$\therefore$ The rate of surface is $1.27 cm^2 s^{-1}$

Exactly as expected, like assessment schedule.