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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

National Certificate of Educational Achievement
TAUMATA MĀTAURANGA Ā-MOTU KUA TAEA

Level 3 Calculus, 2006

90635 Differentiate functions and use derivatives to solve problems

Credits: Six
9.30 am Wednesday 29 November 2006

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Make sure you have a copy of the Formulae and Tables booklet L3-CALCF.

You should answer ALL the questions in this booklet.

Show ALL working for ALL questions.

Show any derivatives that you need to find when solving the problems.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

For Assessor's use only	Achievement Criteria	
Achievement	**Achievement with Merit**	**Achievement with Excellence**
Differentiate functions and use derivatives to solve problems. ✓	Demonstrate knowledge of advanced concepts and techniques of differentiation and solve differentiation problems. —	Solve more complex differentiation problem(s). —
Overall Level of Performance		A.

Interesting to note in teacher feedback that teachers have not noted change in version 2 of the standard that simple product rule and quotient rule problems are now achievement level not merit level as they were in version 1.

Exemplar ACHIEVEMENT

Good accurate differentiation skills. Scored 3/3 code A1s.

Could differentiate

1) A simple composite function with the chain rule
2) A simple trig function with the chain rule
3) A simple quotient where the chain rule was <u>not</u> required.

You are advised to spend 50 minutes answering the questions in this booklet.

QUESTION ONE

Differentiate the following functions.
You do not need to simplify your answers.

(a) $y = (x^2 - 3x)^5$

$\cancel{\frac{dy}{dx} = -3 \cdot 2x \cdot 5(x^2 - 3x)^4}$ $\quad \frac{dy}{dx} = (2x-3) \cdot 5(x^2 - 3x)^4$

$\cancel{= -30x(x^2 - 3x)^4}$

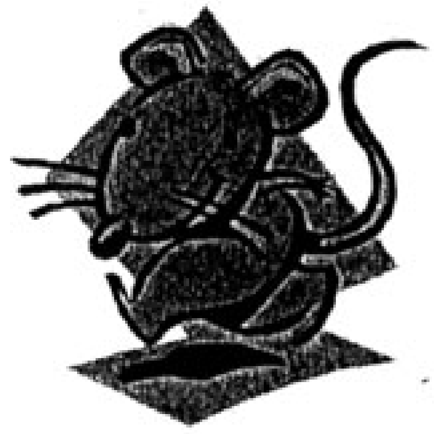
(b) $y = 5\cot 2x$

$\frac{dy}{dx} = -10\operatorname{cosec}^2 2x$

(c) $y = \dfrac{\sin x}{x+3}$

$\frac{dy}{dx} = \dfrac{(x+3)\cdot \cos x - \sin x \cdot 1}{(x+3)^2}$

QUESTION TWO



A teenager breeds mice for pet shops.
The number of mice for the first nine months of his production can be modelled by:

$N(t) = 10e^{0.5t} + 12\ln(2t+7),\ 0 \le t \le 9$

where N is the total number of mice
and t is the time in months.

At what rate is the number of mice increasing at 7 months?

Show any derivatives that you need to find when solving this problem.

$\frac{dN}{dt} = 5e^{0.5t} + \frac{12\times 2}{2t+7}$

when $t = 7$

$\frac{dN}{dt} = 5e^{0.5\cdot 7} + \frac{12\times 2}{21}$

$= \cancel{58.28}$ (4sf)

58.85

$\frac{dN}{dt}$ is correct.

$\frac{dN}{dt} = 5e^{0.5t} + \frac{24}{2t+7}$

To achieve the standard, the candidate needs at least one of code A2, that means they can solve a problem after finding the correct derivative.

Here the candidate has the correct derivative in Q2 but has not been able to find the correct value for $\frac{dN}{dt}$ when $t=7$

Correct use of calculator would have resulted in 166.7/167 mice

QUESTION THREE

The power of an engine of a sports car can be modelled by the function:

$$P(x) = 480 - \frac{500\,000}{x} - \frac{x}{25}, \; 1200 \le x \le 6000$$

where P is the power of the engine (in kilowatts)
and x is the speed of the engine (in revolutions per minute).

Calculate the speed of the engine that generates the maximum power.

You may assume $\frac{d^2P}{dx^2} < 0$.

Show any derivatives that you need to find when solving this problem.

$\cancel{\frac{dP}{dx} = +500000x^{-2} - \frac{x^2}{25}}$

$\cancel{0 = 500000x^{-2} - \frac{x^2}{25}}$

$\cancel{x = 12500000x^{-2} - x^2}$

$\cancel{x^2 \cdot x^2 = 12500000}$

$\cancel{x^4 = 12500000}$

$\cancel{x = \pm\sqrt[4]{12500000}}$

$P(x) \frac{dP}{dx} = 480 - 500000x^{-1} - \frac{x}{25}$

$0 = 500000x^{-2} - \frac{1}{25}$

$0 = \frac{12500000}{x^2} - 1$

$x^2 = 12500000$

$x = \pm 3535.53$ (-3535.53 not possible because ($1200 \le x \le 6000$)) A2

$x_{max} = 3535.53$

$P_{max} = 480 - \frac{500000}{3535.53} - \frac{3535.53}{25}$

$= 197.16$ (5 s.f.)

The candidate has achieved the second of the code A2s by showing that they could rewrite the function with a negative power, differentiate to find $P'(x)$ correctly and then knew they had to solve $P'(x) = 0$ to find x when the power of the engine is at a maximum.

They demonstrated good algebra skills to achieve this rather than relying on their graphic calculator

They have done more than the question required by continuing to find P_{max}.

QUESTION FOUR

Use the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to show that the derivative of $f(x) = 3x^2 + x + 5$ is $f'(x) = 6x + 1$.

~~$f'(x) = \lim_{h\to 0} f(3x^2+x+5+h) - (3x^2+x+5)$~~

$f'(x) = \lim_{h\to 0} \dfrac{f(3(x+h)^2 + x+h+5) - 3x^2+x+5}{h}$

$= \lim \dfrac{3(x^2+2xh+h^2)+x+h+5 - 3x^2+x+5}{h}$

$= \lim \cancel{3x^2} + 6xh + 3h^2 + \cancel{x} + h \cancel{+5} - (\cancel{3x^2} \cancel{+x} \cancel{+5})$

$= \lim \dfrac{6x\cancel{h} + 3h^{\cancel{2}} + \cancel{h}}{\cancel{h}}$

$= \lim 6x + 3h + 1$

when $h = 0$

$= 6x + 1$

To achieve merit they needed three code Ms Question 4 and 5 were assessing knowledge of differentiation concepts and Q6 and Q7 were assessing the candidate's ability to solve a differentiation problem, harder than those in Q2 and Q3.

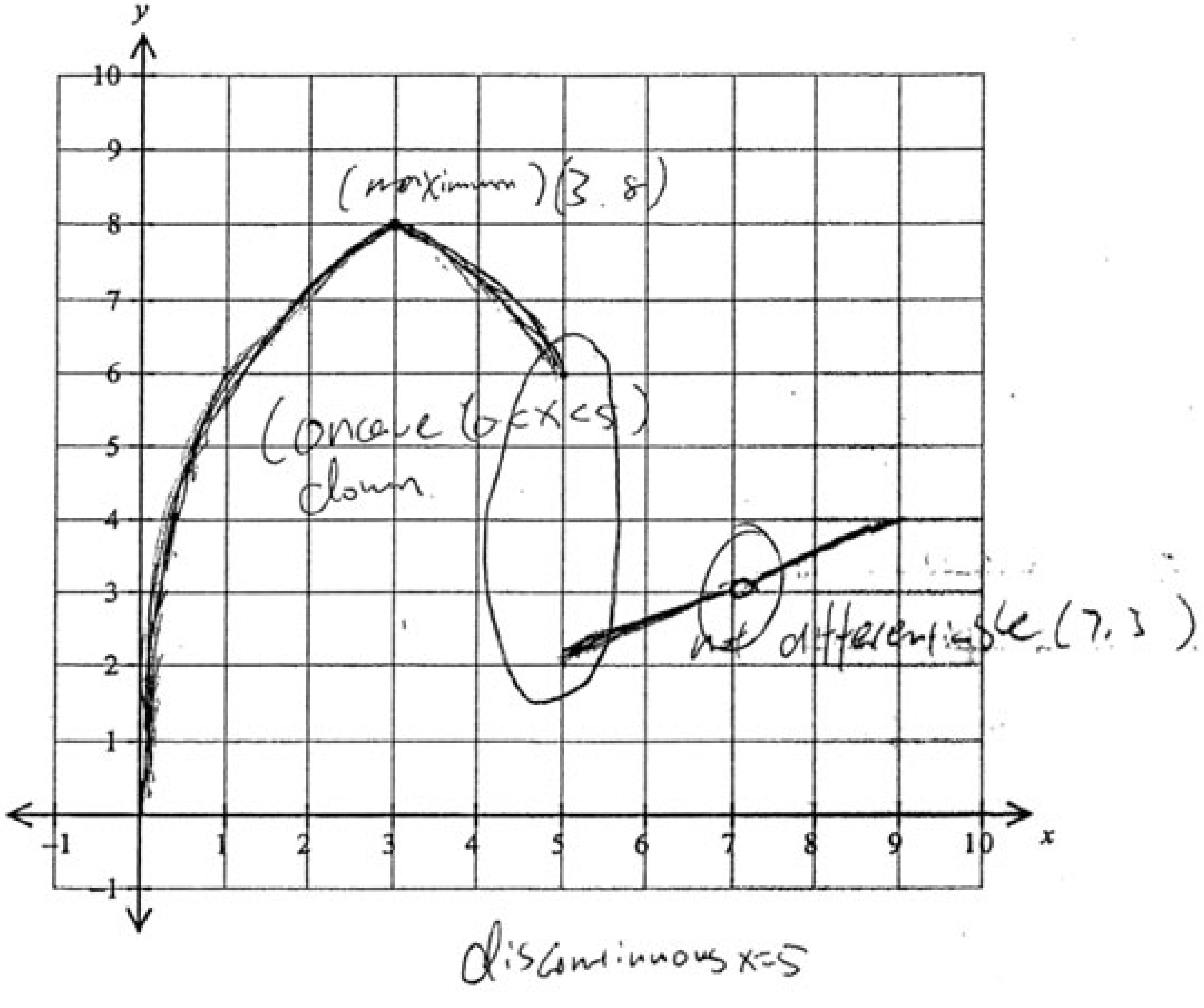
Q4 scored N.
The setting out was sloppy - the candidate failed to write the appropriate limit statement 4 times but more importantly ~~t~~ they did not multiply everything in the bracket by 3 so there is an error that continues through 3 lines of the working.

QUESTION FIVE

On the axis below, sketch a graph of $f(x)$ that

- is continuous for $0 < x < 5$ and $5 < x < 9$ and is discontinuous when $x = 5$ X
- is concave down ($f''(x) < 0$) for $0 < x < 5$
- has $f'(x) = 0$ at (3,8)
- has $\lim_{x \to 5} f(x) = 6$ X
- is not differentiable at (7,3).

If you need to redraw this graph, use page 10.



N

Q5 Scored N.

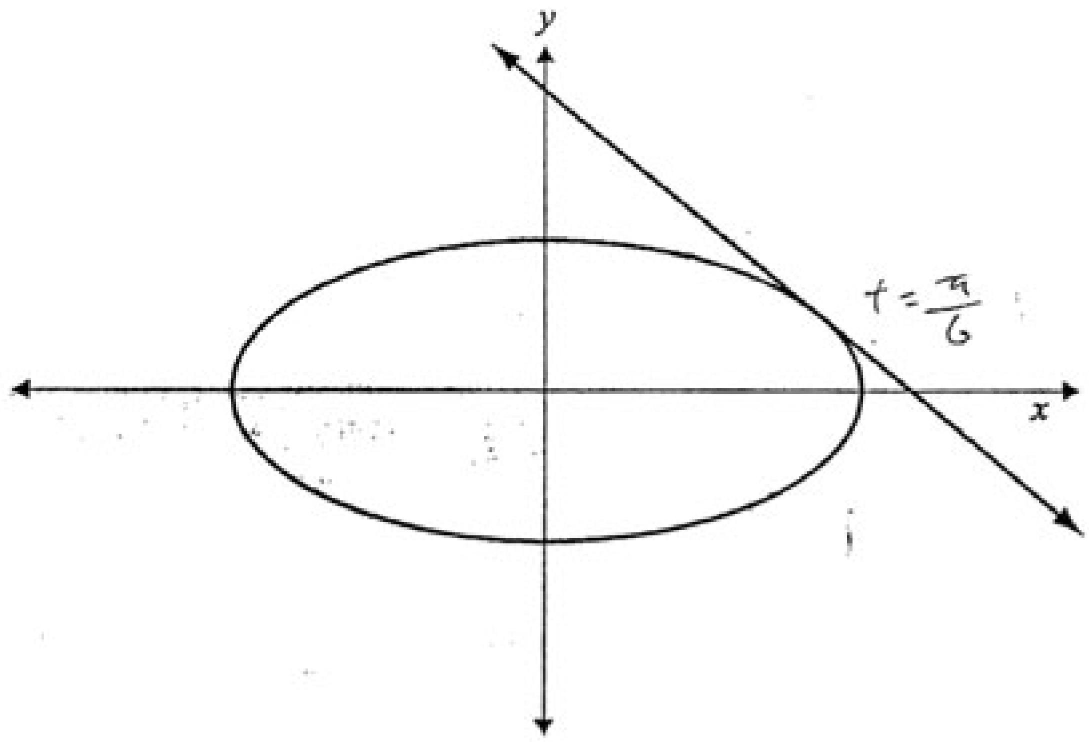
The candidate's graph fails on 2 of the 5 criteria listed. It is not continuous for $5<x<9$. Also for the graph drawn there is no limit for the function $f(x)$ when $x=5$.

The candidate has demonstrated they do not understand limit statements nor how to achieve a point on the graph where the function is not differentrable but still continuous - very important merit level differentiation concepts.

QUESTION SIX

The graph below shows the curve defined by the parameters $x = 6\cos t$ and $y = 4\sin t$.

It also shows the tangent to the curve at the point $t = \frac{\pi}{6}$.



Find the y-intercept of the tangent to the curve when $t = \frac{\pi}{6}$.

Show any derivatives that you need to find when solving this problem.

$\frac{dx}{dt} = \cancel{6\cos} - 6\sin t$

$\frac{dy}{dt} = 4\cos t$

$\frac{dy}{dx} = 4\cos t \cdot \frac{1}{-6\sin t} = \frac{4\cos t}{-6\sin t}$

$m = \frac{4\cos\frac{\pi}{6}}{-6\sin\frac{\pi}{6}}$

$= -1.15$

$y - 2 = -1.15(x - 8.2)$

$y = -1.15x + 5.98 + 2$

$y = -1.15x + 7.98$

when $x = 0$

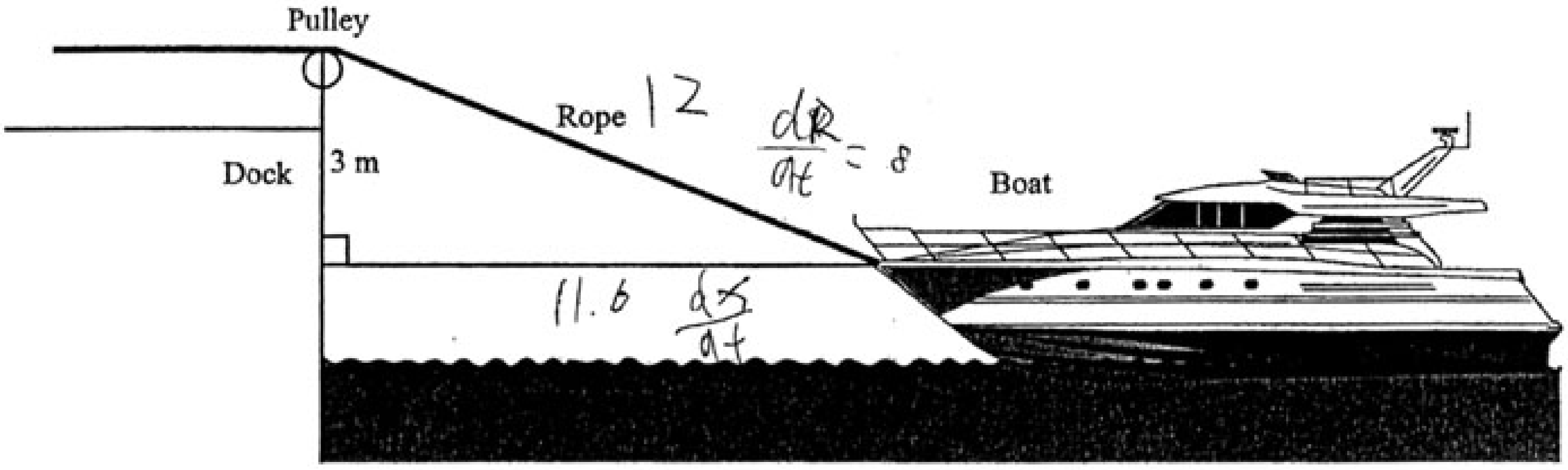
$y = 7.98.$

The correct answer for Q6 is
$y = 8$.
This candidate is not perfectly correct with $y = 7.98$ but they have been marked M2 because they have correct supporting working. $\frac{dy}{dx} = \frac{4\cos t}{-6\sin t}$

Accuracy has been lost by rounding to 2 decimal places instead of using surds.

QUESTION SEVEN

A boat is pulled into dock by means of a rope running through a pulley on the dock.
The rope is attached to the bow of the boat at a point 3 metres below the level of the pulley.
The rope is being pulled through the pulley at a rate of 8 metres per minute.



At what rate will the boat be approaching the dock when there is 12 metres of rope between the boat and the pulley?

Show any derivatives that you need to find when solving this problem.

$R^2 - 3^2 = x^2$

$\frac{dR}{dx} = \frac{2x}{2R}$ $\quad$ $R = 12$

$= \frac{x}{R=12} = \frac{x}{12}$

$\frac{dR}{dt} = 8$

$\frac{dx}{dt} = \frac{dx}{dR} \cdot \frac{dR}{dt}$

$= \frac{12}{x} \cdot 8$ m/min

$x = 11.6$ m

$\frac{dx}{dt} = 8.276$ m/min

This is a very good response. Most students would have ~~[illegible]~~ rearranged the pythagoras statement before differentiating but this student has differentiated implicitly and substitutely correctly to achieve a final answer of 8.26 m/min.

QUESTION EIGHT

A spherical balloon is being filled with air at a constant rate.
At a certain time, the volume is 360 cm^3.
20 seconds later, the volume is 450 cm^3.

Find the rate of increase of the surface area when the balloon has a volume of 1500 cm^3.

Show any derivatives that you need to find when solving this problem.

$V = \frac{dV}{dt} \cdot t$ $\qquad 450 = \frac{dV}{dt} \cdot 20 + t$

$360 = \frac{dV}{dt} \cdot t$

$450 - 360 = \frac{dV}{dt} \cdot 20$

$\frac{dV}{dt} = 4.5$

The student has made a good start by using the information provided to calculate $\frac{dV}{dt} = 4.5 \ cm^3 s^{-1}$

They needed to keep going to write an expression for $\frac{dA}{dt}$, find r when $V = 1500 \ cm^3$ and hence $\frac{dA}{dt}$ when $V = 1500 \ cm^3$

Interesting candidate
Not common to see students doing so well on problem-solving in preference to the straight skill type questions.

More attention to detail in Q4 and Q5 would have allowed this student to achieve merit.