For a particle of mass m and velocity $\mathbf{v}$ be located at a position $\mathbf{r}$ measured from some origin. Then according to classical mechanics, the particle has a linear momentum $\mathbf{p}$ given by

$$\mathbf{p} = m\mathbf{v}$$

and an angular momentum $\boldsymbol{\ell}$ given by

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$$

We transcribe this to a quantum representation by replacing $\mathbf{p}$ with $(\hbar/i)\nabla$ where

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

In Equation (1.3) (and elsewhere) a superscript caret denotes a unit vector. For convenience, we drop the burden of carrying around $\hbar$ by introducing a system of units in which $\hbar = 1$. Thus the Cartesian components of $\mathbf{p}$ are

$$p_x = -i\frac{\partial}{\partial x}, \qquad p_y = -i\frac{\partial}{\partial y}, \qquad p_z = -i\frac{\partial}{\partial z}$$

and those of $\boldsymbol{\ell}$ are

$$l_x = yp_z - zp_y = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$l_y = zp_x - xp_z = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$
$$l_z = xp_y - yp_x = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

The commutator $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ of two operators $\mathbf{A}$ and $\mathbf{B}$ plays a central role in quantum mechanics; the necessary condition for the observables A and B to be simultaneously measurable is that the corresponding operators $\mathbf{A}$ and $\mathbf{B}$ commute, that is, $[\mathbf{A},\mathbf{B}] = 0$. From equation (1.4) it is easily seen that the position vector of a particle and its momentum satisfy the basic commutation relations:

$$[x, p_x] = i, \qquad [x, p_y] = 0, \qquad [x, p_z] = 0$$

with all cyclic permutations. Thus, for example it is not possible to measure simultaneously along the same direction the position and linear momentum of a particle to arbitrary precision.

The commutation relations of the Cartesian components of $\boldsymbol{\ell}$ are also readily derived:

$$[l_x, l_y] = il_z, \qquad [l_y, l_z] = il_x, \qquad [l_z, l_x] = il_y$$

Equation (1.7) has the interpretation that quantum states cannot be specified by any more than one of the labels (eigenvalues) of the three components of angular momentum. The "good" quantum numbers corresponding to the largest set of mutually commuting operators represent the maximum information that can be known about a quantum mechanical system. The measurement of another variable corresponding to an operator not commuting with this set necessarily introduces uncertainty into one of the variables already measured. A sharper specification of the system is, therefore, not possible.

Because of the importance of the commutator, it is natural to define a general angular momentum operator **j** as one whose Cartesian components obey the commutation rules

$$\left[j_x, j_y \right] = ij_z, \qquad \left[j_y, j_z \right] = ij_x, \qquad \left[j_z, j_x \right] = ij_y \tag{1.8}$$

in analogy to equation (1.7). This extended definition returns an unexpected dividend. As we shall dee in the next section, it permits the existence of spin—a quantity that has no classical analogy. We will reserve ℓ for orbital angular moment and use **j** for general angular momentum.

Eigenvalues and matrix elements of angular momentum operators

The square of the total angular momentum is defined as

$$\mathbf{j}^2 = j_x^2 + j_y^2 + j_z^2 \tag{1.9}$$

This operator has the commutation properties that

$$\left[\mathbf{j}^2, j_x \right] = \left[\mathbf{j}^2, j_y \right] = \left[\mathbf{j}^2, j_z \right] = 0 \tag{1.10}$$

Hence we may construct states $|\, jm\rangle$ that are simultaneously eigenfunctions of $\mathbf{j}^2$ and any one component of **j**, say j_z; that is

$$\begin{aligned} \mathbf{j}^2 \,|\, jm\rangle &= \lambda_j \,|\, jm\rangle \\ j_z \,|\, jm\rangle &= m \,|\, jm\rangle \end{aligned} \tag{1.11}$$

We proceed to determine the eigenvalues $\lambda_j = \langle jm |\, \mathbf{j}^2 \,|\, jm\rangle$ and $m = \langle jm |\, j_z \,|\, jm\rangle$.

The operator $j_x^2 + j_y^2 = \mathbf{j}^2 - j_z^2$ is diagonal in the $|\, jm\rangle$ representation. Moreover it has positive definite (nonnegative) eigenvalues.

$$\begin{aligned} \left(j_x^2 + j_y^2 \right) |\, jm\rangle &= \left(\mathbf{j}^2 - j_z^2 \right) |\, jm\rangle \\ &= \mathbf{j}^2 \,|\, jm\rangle - j_z^2 \,|\, jm\rangle \\ &= \lambda_j \,|\, jm\rangle - j_z \left(m |\, jm\rangle \right) = \lambda_j \,|\, jm\rangle - m\left(j_z \,|\, jm\rangle \right) \\ &= \lambda_j \,|\, jm\rangle - m^2 \,|\, jm\rangle \\ &= \left(\lambda_j - m^2 \right) |\, jm\rangle \end{aligned} \tag{1.12}$$

because the expectation value of the square of a Hermitian operator, that is, the square of a real eigenvalue, is greater than or equal to zero. Hence we conclude that the value of m is bounded from both above and below in that m^2 cannot exceed λ_j. This implies that for a given **j** there exist minimum and maximum values of m, denoted by $m_{\min}$ and $m_{\max}$, respectively.

Now we introduce the raising and lowering operators $j_\pm$ defined by

$$j_+ = j_x + ij_y \qquad j_- = j_x - ij_y \tag{1.13}$$

From equations (1.8) and (1.10) it may be shown that these operators satisfy the commutation rules:

$$\begin{aligned}\left[\mathbf{j}^2, j_\pm\right] &= 0 \\ \left[j_z, j_\pm\right] &= \pm j_\pm \\ \left[j_+, j_-\right] &= 2 j_z\end{aligned} \tag{1.14}$$

The commutation algebra is useful here:

$$\begin{aligned}[\mathbf{A},\mathbf{B}] &= \\ [\mathbf{A}, b\mathbf{B}] &= \\ [\mathbf{AB},\mathbf{C}] &= \\ \left[\mathbf{A},\mathbf{B}+\mathbf{C}\right] &= \\ \left[\mathbf{A}+\mathbf{B},\mathbf{C}+\mathbf{D}\right] &= \\ \left[\mathbf{A},\left[\mathbf{B},\mathbf{C}\right]\right] &= \end{aligned} \tag{1.15}$$

Let us examine the behaviour of the function $j_\pm | jm\rangle$. We find

$$\mathbf{j}^2 j_\pm | jm\rangle = j_\pm \mathbf{j}^2 | jm\rangle = \lambda_i j_\pm | jm\rangle \tag{1.16}$$

and

$$\begin{aligned} j_z j_\pm | jm\rangle &= \left(j_\pm j_z \pm j_\pm\right) | jm\rangle = j_\pm j_z | jm\rangle \pm j_\pm | jm\rangle \\ &= m j_\pm | jm\rangle \pm j_\pm | jm\rangle = (m \pm 1)\, j_\pm | jm\rangle \end{aligned} \tag{1.17}$$

Thus $j_\pm | jm\rangle$ is an eigenfunction of $\mathbf{j}^2$ with the eigenvalue λ_j and an eigenfunction of j_z with the eigenvalue $m \pm 1$. It follows that $j_\pm | jm\rangle$ is proportional to the normalised eigenfunction $| j\ m \pm 1\rangle$. That is

$$j_\pm | jm\rangle = C_\pm | j\ m \pm 1\rangle \tag{1.18}$$

where $C_\pm$ is a proportionality constant. The ability of the raising and lowering operators $j_\pm$ to alter m by ± 1 unit while preserving λ_j gives them their names. The may also be referred to as *step-up* and *step-down*, *ladder* and *shift* operators.

Since the values of m are bounded between m_{min} and m_{max}, it follows that

$$j_+ | j\ m_{\text{max}}\rangle = 0 \tag{1.19}$$

and

$$j_- | j\ m_{\text{min}}\rangle = 0 \tag{1.20}$$

We now apply j_- to equation (1.19) and j_+ to equation (1.20) and by using the identity

$$j_\mp j_\pm = \mathbf{j}^2 - j_z\left(j_z \pm 1\right) \tag{1.21}$$

we obtain the simultaneous equations

$$\begin{aligned}\lambda_j - m_{\text{max}}\left(m_{\text{max}} + 1\right) &= 0 \\ \lambda_j - m_{\text{min}}\left(m_{\text{min}} + 1\right) &= 0\end{aligned}. \tag{1.22}$$

Eliminating λ_j yields

$$m_{\max}\left(m_{\max}+1\right)=m_{\min}\left(m_{\min}-1\right) \tag{1.23}$$

or

$$\left(m_{\max}+m_{\min}\right)\left(m_{\max}-m_{\min}+1\right)=0 \tag{1.24}$$

One of these two factors must vanish. We know, however, that $m_{\max} \geq m_{\min}$ so the only solution to equation (1.23) is

$$m_{\max}=-m_{\min} \tag{1.25}$$

Successive values of *m* differ by unity (shown in equation (1.17)). Therefore, $m_{\max}-m_{\min}$ is a positive definite integer which we may denote with 2*j*, where *j* is an integer or half integer. Then from $m_{\max}-m_{\min}=2j$ and $m_{\max}+m_{\min}=0$ we conclude that

$$m_{\max}=j, \qquad m_{\min}=-j \tag{1.26}$$

and there are 2*j*+1 possible values of *m*, $m=j, j-1, j-2, \ldots, -j+1, -j$ for each value of *j*. Substitution of equation (1.26) into equation (1.22) yields the additional result

$$\lambda_j=j\left(j+1\right) \tag{1.27}$$

We are also in a position to evaluate the proportionality constant $C_{\pm}$ appearing in equation (1.18). We find

$$\begin{aligned}\left|C_{\pm}\right|^2 &=\left\langle jm\right| j_{\mp} j_{\pm}\left| jm\right\rangle=\left\langle jm\right| \mathbf{j}^2-j_z\left(j_z \pm 1\right)\left| jm\right\rangle \\ &=j\left(j+1\right)-m\left(m \pm 1\right)\end{aligned} \tag{1.28}$$

From this we can see that the absolute value of $C_{\pm}$ is determined but the phase is arbitrary (it could be positive or negative). We choose $C_{\pm}$ to be real, that is

$$C_{\pm}=\left[j\left(j+1\right)-m\left(m \pm 1\right)\right]^{\frac{1}{2}} \tag{1.29}$$

This agrees with the standard phase convention namely that the matrix elements of j_x are real while those of j_y are purely imaginary.

We have here a treatment in which no explicit function had been used. Our starting assumption that the system was not classical is sufficient to bring in quantisation and to set limits on the values which the quantum numbers may take.