Lecture 5

Actually, some differential equations (Bessel's equation, for instance) have solutions of a more general form, namely, power series times a single power of x

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n. \tag{1.2}$$

We will show how the Power Series Method can be generalized to such equations. The generalized Power Series Method is known as

Frobenius Method.

[*Reading: EK, Chapter 4*]

Power Series

DEFINITION 12 A power series (in powers of $x - x_0$) is an infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ..., \tag{1.3}$$

where a_n are coefficients of the series, x_0 is called the centre of the series and x is a variable.

In the following we take $x_0 = 0$. This does not mean that we consider only a particular case because we always can treat $x - x_0$ as a new variable, say, ξ. In addition, we shall assume that all variables and constants are real.

EXAMPLE:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \quad (|x| < 1) \tag{1.4}$$

[*Check by multiplying both sides by* $1 - x$.]

EXAMPLE:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{1.5}$$

This formula follows from the Taylor series expansion for a given function $f(x)$ at the point $x = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{1.6}$$

Here $f^{(n)}(0)$ denotes the n-th derivative of the function $f(x)$ at $x = 0$ and $n! = 1 \times 2 \times 3 ...... \times n$.

[*Note that* $\frac{d^n}{dx^n}[e^x] = e^x$ *and, therefore,* $f^{(n)}(0) = 1$.]

Convergence of power series (1.3) is understood in the usual sense:

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n x^n \quad \text{exists.}$$

The series (1.3) defines a function $y(x)$ on interval $I = (x_0 - R, x_0 + R)$ ($I = (-R, R)$ for $x_0 = 0$), where R is the radius of convergence given by the formula:

$$\frac{1}{R} = \lim_{n \to \infty} |a_n|^{\frac{1}{n}} \tag{1.7}$$

or

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{1.8}$$

For the power series (1.4), $a_n = 1$ and formula (1.8) gives $R = 1$.

For the power series (1.5), $a_n = 1/n!$ and formula (1.8) gives $R = \infty$.

IMPORTANT Most of the elementary functions can be presented as power series. A function defined by a power series is continuous and differentiable in I. The derivative can be computed by differentiating the series term by term. The resulting power series for the derivative has the same radius of convergence R. Therefore, any derivative of the function can be computed. The power series can be integrated term-by-term within the interval of its convergence.

THEOREM 6 [Identity Principle] If

$$\sum_{n=0}^{\infty} a_n x^n = 0 \tag{1.9}$$

for any $x \in I$, then $a_n = 0$ for $n \geq 0$.

PROOF Take $x = 0$ in (1.9) and deduce that $a_0 = 0$. Now the summation in (1.9) starts from $n = 1$. Divide both sides of (1.9) by x, take $x = 0$ and deduce that $a_1 = 0$. Continue the procedure and obtain that $a_n = 0$ for $n \geq 0$.

POWER SERIES METHOD

THEOREM 7 [Fuch's (1833-1902)] Consider the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

with initial conditions

$$y(0) = K_0, \quad y'(0) = K_1.$$

If both $p(x)$ and $q(x)$ have Taylor series, which converge on the interval $I = (-R, R)$, $R > 0$, then the IVP has unique power series solution $y(x)$, which also converges on the interval I.

Radius of convergence of the series solution is at least as big as the minimum of the radii of convergence of $p(x)$ and $q(x)$.

EXAMPLE: Let us consider how to obtain a power series solution for the homogeneous Legendre equation

$$(1 - x^2)y'' - 2xy' + m(m + 1)y = 0 \tag{1.10}$$

or, in standard form,

$$y'' - \frac{2x}{1 - x^2}y' + \frac{m(m + 1)}{1 - x^2}y = 0, \tag{1.11}$$

where m is integer.

In Fuch's theorem,

$$p(x) = -\frac{2x}{1 - x^2} = -2\sum_{n=0}^{\infty} x^{2n+1}, \quad q(x) = m(m + 1)\sum_{n=0}^{\infty} x^{2n}$$

with radius of convergence $R = 1$, see (1.4). Therefore, equation (1.11) (and correspondingly, equation (1.10)) has two linearly independent solutions in the form of power series, see Theorem 5.

The power series solution has the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{1.12}$$

Substitute (1.12) into (1.10)

$$(1 - x^2)\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + m(m + 1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

By algebra

$$\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} + \sum_{n=0}^{\infty}[-n(n - 1) - 2n + m(m + 1)]a_n x^n = 0. \tag{1.13}$$

In the first term, we introduce $n-2=k$ which gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n. \tag{1.14}$$

Equations (1.13) and (1.14) provide

$$\sum_{n=0}^{\infty}\{(n+2)(n+1)a_{n+2} - [n(n-1)+2n-m(m+1)]a_n\}x^n = 0 \quad (-1<x<1).$$

Identity Principle gives

$$(n+2)(n+1)a_{n+2} - [n(n-1)+2n-m(m+1)]a_n = 0 \quad (n \geq 0)$$

and

$$a_{n+2} = \frac{n^2+n-m^2-m}{(n+1)(n+2)}a_n = \frac{(n-m)(n+m+1)}{(n+1)(n+2)}a_n \quad (n \geq 0). \tag{1.15}$$

If $a_0 = 0$ and $a_1 \neq 0$, then the even coefficients a_{2k} are zero, according to (1.15,) and the solution is

$$y_1(x) = a_1 x + a_3 x^3 + \ldots\ldots\ldots\ldots. \tag{1.16}$$

where

$$a_3 = \frac{(1-m)(2+m)}{1\cdot 2\cdot 3}a_1, \quad a_5 = \frac{(3-m)(4+m)}{4\cdot 5}a_3 = \frac{(1-m)(3-m)(2+m)(4+m)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a_1, \ldots\ldots\ldots$$

Lecture 6

If $a_1 = 0$ and $a_0 \neq 0$, then the odd coefficients a_{2k+1} are zero, according to (1.15,) and the solution is

$$y_2(x) = a_0 + a_2 x^2 + \ldots\ldots\ldots\ldots. \tag{1.17}$$

where

$$a_2 = \frac{-m(1+m)}{1\cdot 2}a_0, \quad a_4 = \frac{(2-m)(3+m)}{3\cdot 4}a_2 = \frac{(0-m)(2-m)(1+m)(3+m)}{1\cdot 2\cdot 3\cdot 4}a_0, \ldots\ldots\ldots$$

If $m=0$, then $a_2 = 0$, $a_4 = 0$ and so on. Therefore, the power series for $y_1(x)$ consists of infinite number of terms but $y_2(x) = a_0$.

If $m=1$, then $a_3 = 0$, $a_5 = 0$ and so on. Therefore, $a_{2k+1} = 0$ for $k \geq 1$. In this case, the power series for $y_2(x)$ consists of infinite number of terms but $y_1(x) = a_1 x$.

If $m = 2$, then $a_2 = -3a_0$, $a_4 = 0$ and so on. Therefore, $y_1(x)$ is given by infinite series but

$$y_2(x) = a_0 + a_2x^2 = a_0 - 3a_0x^2 = a_0(1 - 3x^2).$$

Solutions $y_1(x)$ and $y_2(x)$ are linearly independent (*Why?*) and a general solution of (1.10) is

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are given by power series (1.16) and (1.17), correspondingly. These power series converge on the interval $(-1, 1)$.

However, they do not converge at $x = \pm 1$!!!!!!!!

At the end of Lecture 3 it was shown that equation (1.10) with $m = 1$ has two solutions

$$y_1(x) = x, \quad y_2(x) = -1 + \frac{x}{2}\log\Big(\frac{1+x}{1-x}\Big).$$

It is seen that $y_2(x)$ is singular at $x = \pm 1$ but $y_1(x)$ is regular.

IMPORTANT A regular solution of Legendre equation (1.10) exists if and only if the number of terms in (1.16) or (1.17) is finite.

If m is even, then the solution (1.16) is singular at $x = \pm 1$ but the solution (1.17) is regular.

If m is odd, then the solution (1.17) is singular at $x = \pm 1$ but the solution (1.16) is regular.

IMPORTANT If m is not integer, then there is no regular solution of (1.10).

LEGENDRE POLYNOMIALS

DEFINITION 13 For integer m a regular solution of (1.10) which is equal to unity at $x = 1$ is known as the Legendre polynomial, $P_m(x)$, of degree m.

EXAMPLE:

$m = 0$: $\quad y_2(x) = a_0, \quad a_{2n} = 0 \ (n \geq 1); \quad y_2(1) = 1 \implies a_0 = 1 \implies \boxed{P_0(x) = 1}$

$m = 1$: $\quad y_1(x) = a_1x, \quad a_{2n+1} = 0 \ (n \geq 1); \quad y_1(1) = 1 \implies a_1 = 1 \implies \boxed{P_1(x) = x}$

$m = 2$: $\quad y_2(x) = a_0 - 3a_0x^2, \quad a_{2n} = 0 \ (n \geq 2);$

$$y_2(1) = a_0(1 - 3 \cdot 1) = 1 \implies a_0 = -\tfrac{1}{2} \implies \boxed{P_2(x) = \frac{1}{2}(3x^2 - 1)}$$

$m = 3$: $\quad y_3(x) = a_1x - \tfrac{5}{3}a_1x^3, \quad y_3(1) = a_1(1 - \tfrac{5}{3} \cdot 1) = -\tfrac{2}{3}a_1 = 1 \implies$

$$a_1 = -\tfrac{3}{2} \implies \boxed{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

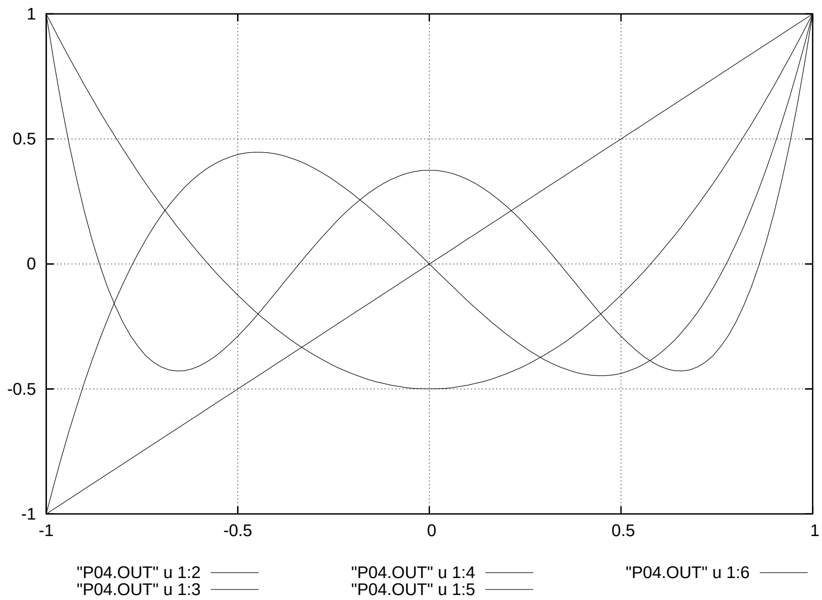


Figure 1. Legendre polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$.

It is not easy to calculate the Legendre polynomials of higher degrees in this way.

It is more convenient to use Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2-1)^n]. \tag{1.18}$$

We do not prove formula (1.18) in this module but we demonstrate how it works

$$m = 0: \qquad P_0(x) = \frac{1}{2^0 0!} \frac{\mathrm{d}^0}{\mathrm{d}x^0}[(x^2-1)^0] = 1.$$

$$m = 1: \qquad P_1(x) = \frac{1}{2^1 1!} \frac{\mathrm{d}^1}{\mathrm{d}x^1}[(x^2-1)^1] = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x}[x^2-1] = x$$

$$m = 2: \qquad P_2(x) = \frac{1}{2^2 2!} \frac{\mathrm{d}^2}{\mathrm{d}x^2}[(x^2-1)^2] = \frac{1}{4 \cdot 1 \cdot 2} \frac{\mathrm{d}}{\mathrm{d}x}[2(x^2-1) \cdot 2x] = \frac{1}{2}(3x^2-1)$$

Calculate $P_4(x)$ using Rodrigues' formula (1.18).

Comment Sometimes the Legendre polynomials are defined by (1.18).

The Legendre polynomials can also be calculated step by step using the following recurrence relation

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad (n = 1, 2, 3, ...) \tag{1.19}$$

EXAMPLE:

You are given $P_0(x) = 1$ and $P_1(x) = x$. Derive formulae for the Legendre polynomials using (1.19).

$$n = 1: \qquad 2 \cdot P_2(x) - 3xP_1(x) + 1 \cdot P_0(x) = 0 \implies 2P_2(x) - 3x^2 + 1 = 0, \;\; P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$n = 2: \qquad 3 \cdot P_3(x) - 5xP_2(x) + 2 \cdot P_1(x) = 0 \implies 3 \cdot P_3(x) - \frac{5}{2}x(3x^2 - 1) + 2x = 0, \;\; P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

EXAMPLE:

Calculate $P_4(0)$.

Take $n = 3$ in (1.19).

$$4P_4(0) - 0 + 3P_2(0) = 0, \quad \implies P_4(0) = -\frac{3}{4}P_2(0) = -\frac{3}{4} \cdot \frac{1}{2}(3 \cdot 0^2 - 1) = \frac{3}{8}.$$

Derivatives of the Legendre polynomials can be calculated using the following recurrence relation

$$P'_{n+1}(x) = P'_{n-1}(x) + (2n+1)P_n(x). \tag{1.20}$$

EXAMPLE:

Calculate $P'_3(x)$ by differentiating the formula for $P_3(x)$ and by (1.20).

DEFINITION 14 A system of functions $\varphi_n(x)$, $(n = 0, 1, 2, ...)$, is said to be <u>orthogonal</u> on the interval (a, b) [or the functions $\varphi_n(x)$ are said to be orthogonal] if

$$\int_a^b \varphi_n(x)\varphi_m(x)\mathrm{d}x = 0 \quad (n \neq m).$$

DEFINITION 15 A system of functions $\varphi_n(x)$, $(n = 0, 1, 2, ...)$, is said to be <u>orthonormal</u> on the interval (a, b) if the system is orthogonal and

$$\int_a^b \varphi_n^2(x)\mathrm{d}x = 1.$$

THEOREM 8 The Legendre polynomials $P_n(x)$, $n \geq 0$, are orthogonal on the interval $(-1, 1)$.

PROOF Subtract the differential equation (1.10) for $P_n(x)$ multiplied by $P_m(x)$ from (1.10) for $P_m(x)$ multiplied by $P_n(x)$:

[Hint: $[(1-x^2)P_n']' = (1-x^2)P_n'' - 2xP_n'$ and compare with (1.10)]

$$[(1-x^2)P_m']'P_n - [(1-x^2)P_n']'P_m + m(m+1)P_mP_n - n(n+1)P_nP_m = 0$$

or

$$[(1-x^2)\{P_m'P_n - P_n'P_m\}]' + (m-n)(m+n+1)P_mP_n = 0$$

Integrate the latter equation over the interval $(-1, 1)$ and note that the integral of the first term vanishes. Therefore,

$$(m-n)(m+n+1)\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0$$

and

$$\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0 \quad (n \neq m). \tag{1.21}$$

Lecture 7

THEOREM 9 A system of functions $\sqrt{n+\frac{1}{2}}P_n(x)$, $(n = 0, 1, 2, ...)$, is orthonormal on the interval $(-1, 1)$.

PROOF The system of functions $\sqrt{n+\frac{1}{2}}P_n(x)$, $(n = 0, 1, 2, ...)$, is orthogonal on the interval $(-1, 1)$ according to (1.21). We need to show that

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2}{2n+1} \quad (n = 0, 1, 2, ...) \tag{1.22}$$

Replace n by $n-1$ in (1.19) and multiply the result by $(2n+1)P_n(x)$ $[n \geq 2]$

$$n(2n+1)P_n^2(x) - (2n-1)(2n+1)xP_{n-1}(x)P_n(x) + (n-1)(2n+1)P_{n-2}(x)P_n(x) = 0. \tag{a}$$

Multiply (1.19) by $(2n-1)P_{n-1}(x)$

$$(n+1)(2n-1)P_{n+1}(x)P_{n-1}(x) - (2n+1)(2n-1)xP_n(x)P_{n-1}(x) + n(2n-1)P_{n-1}^2(x) = 0$$

and subtract from (a):

$$n(2n+1)P_n^2(x) + (n-1)(2n+1)P_{n-2}(x)P_n(x) - (n+1)(2n-1)P_{n+1}(x)P_{n-1}(x) - n(2n-1)P_{n-1}^2(x) = 0.$$

Integrate this relation over the interval $(-1, 1)$ and use (1.21)

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)\mathrm{d}x \qquad (n \geq 2).$$

Repeated application of this formula gives

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2n-1}{2n+1}\frac{2n-3}{2n-1}\int_{-1}^{1} P_{n-2}^2(x)\mathrm{d}x = \frac{2n-3}{2n+1}\int_{-1}^{1} P_{n-2}^2(x)\mathrm{d}x$$

$$= \frac{2n-3}{2n+1}\frac{2n-5}{2n-3}\int_{-1}^{1} P_{n-3}^2(x)\mathrm{d}x = \frac{2n-5}{2n+1}\int_{-1}^{1} P_{n-3}^2(x)\mathrm{d}x = ... = \frac{2n-(2j-1)}{2n+1}\int_{-1}^{1} P_{n-j}^2(x)\mathrm{d}x.$$

For $j = n-1$ we have

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{3}{2n+1}\int_{-1}^{1} P_1^2(x)\mathrm{d}x = \frac{3}{2n+1}\int_{-1}^{1} x^2\mathrm{d}x = \frac{2}{2n+1}.$$

Expansion of functions in series of Legendre polynomials

Consider a function $f(x)$ defined in the interval $(-1, 1)$.

Is it possible to present this function as a series of Legendre polynomials

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \tag{1.23}$$

and, if yes, how to find the coefficients c_n in (1.23)?
Coefficients c_n are obtained by multiplying (1.23) by $P_m(x)$ and integration the result term by term over the interval $(-1, 1)$. Using (1.21) and (1.22), we find

$$\int_{-1}^{1} f(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}c_m$$

and, therefore,

$$c_n = (n + \frac{1}{2})\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x \qquad (n = 0, 1, 2, ...). \tag{1.24}$$

THEOREM 10 If a function $f(x)$ is piecewise smooth in $(-1, 1)$ and if the integral

$$\int_{-1}^{1} f^2(x)\mathrm{d}x$$

is finite, then the series (1.23), with coefficients c_n given by (1.24), converges to $f(x)$ at every continuity point of $f(x)$ and to

$$\frac{1}{2}[f(x-0) + f(x+0)],$$

if x is a discontinuity point of $f(x)$.

EXAMPLE:

Consider $f(x) = 0$, where $-1 \le x < \alpha$, and $f(x) = 1$, where $\alpha < x \le 1$. According to Theorem 10, this function can be expanded as a series of the form (1.23) with coefficients

$$c_n = (n + \frac{1}{2}) \int_\alpha^1 P_n(x) \mathrm{d}x$$

Integrate (1.20)

$$P'_{n+1}(x) = P'_{n-1}(x) + (2n+1)P_n(x) \quad (n \ge 1)$$

from α to 1 and note that $P_n(1) = 1$:

$$1 - P_{n+1}(\alpha) = 1 - P_{n-1}(\alpha) + (2n+1) \int_\alpha^1 P_n(x) \mathrm{d}x.$$

We find

$$c_n = -\frac{1}{2}[P_{n+1}(\alpha) - P_{n-1}(\alpha)] \quad (n \ge 1), \qquad c_0 = \frac{1}{2}(1 - \alpha).$$

The series has the form

$$f(x) = \frac{1}{2}(1-\alpha) - \frac{1}{2}\sum_{n=1}^{\infty}[P_{n+1}(\alpha) - P_{n-1}(\alpha)]P_n(x), \quad (-1 < x < 1). \tag{1.25}$$

EXAMPLE:

Consider $f(x) = \sqrt{(1-x)/2}$, where $-1 \le x \le 1$. According to Theorem 10, this function can be expanded as a series of the form (details are not given)

$$\sqrt{\frac{1-x}{2}} = \frac{2}{3}P_0(x) - 2\sum_{n=1}^{\infty} \frac{P_n(x)}{(2n-1)(2n+3)}, \quad (-1 < x < 1). \tag{1.26}$$

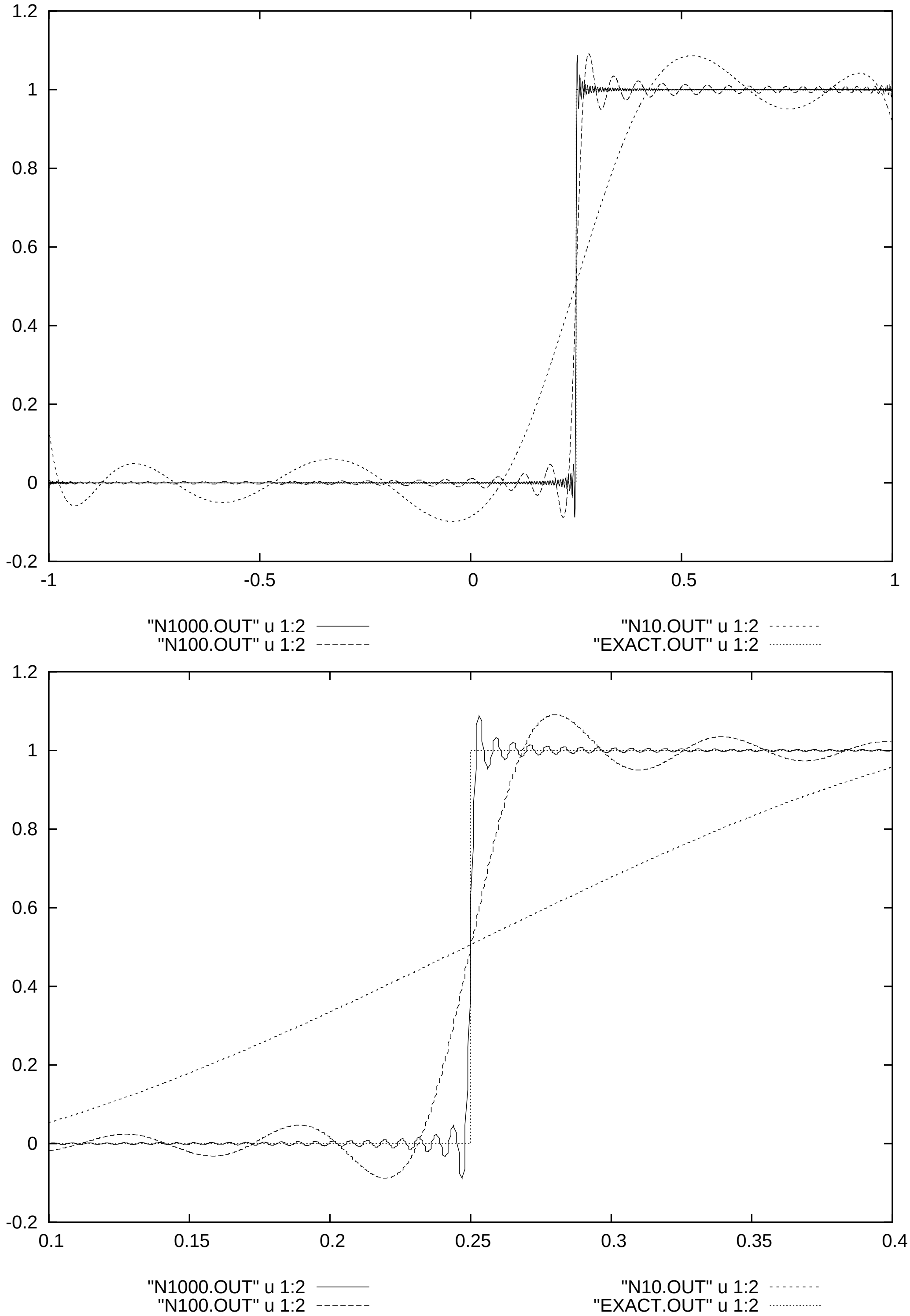


Figure 2. Series of Legendre polynomials (1.25) for $\alpha = 0.25$ with 10, 100 and 1000 terms.

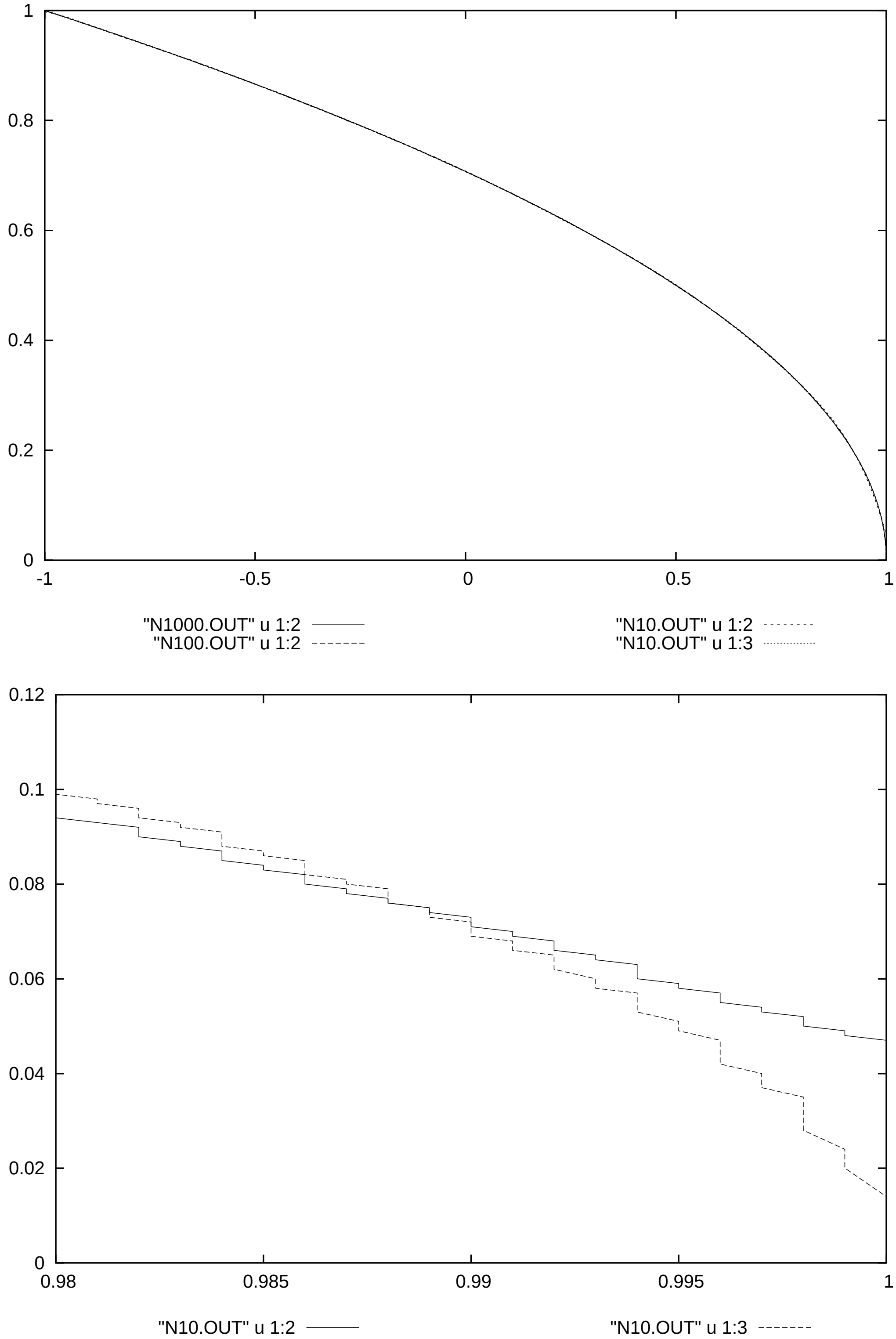


Figure 3. Series of Legendre polynomials (1.26) with 10, 100 and 1000 terms.

Lecture 8

Power Series Method can be used to find solutions of many important differential equations in standard form with coefficients which can be presented by power series.

Consider Airy's equation

$$y'' - xy = 0, \tag{1.27}$$

which is important in theory of surface waves.

Substitute the power series (1.12) into (1.27)

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Replace $n-2$ by m in the first series and $n+1$ by m in the second one

$$2a_2 + \sum_{m=1}^{\infty} \{a_{m+2}(m+2)(m+1) - a_{m-1}\}x^m = 0$$

Identity Principle gives $a_2 = 0$ and

$$a_{m+3} = \frac{a_m}{(m+3)(m+2)}.$$

Consider two cases:

(a) $a_0 \neq 0$, $a_1 = 0$ with the solution

$$y_1(x) = a_0[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + ...];$$

(a) $a_1 \neq 0$, $a_0 = 0$ with the solution

$$y_2(x) = a_1[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + ...]$$

These two solutions are linearly independent (*Why?*) and a general solution of Airy's equation is given as

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

FROBENIUS METHOD

Consider more general form of H-LODE2 than that in the Power series method

$$y'' + \frac{P(x)}{x}y' + \frac{Q(x)}{x^2}y = 0, \tag{1.28}$$

where both $P(x)$ and $Q(x)$ can be represented by their power series

$$P(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad Q(x) = \sum_{n=0}^{\infty} q_n x^n \qquad (|x| < R). \tag{1.29}$$

If $p_0 = 0$, $q_0 = 0$ and $q_1 = 0$, we arrive at the conditions of the Theorem 7 with non-singular coefficients of the differential equation .

DEFINITION 16 The singular point $x = 0$ of the differential equation (1.28) is called regular singular point if the functions $P(x)$ and $Q(x)$ can be represented by their power series (1.29). Otherwise $x = 0$ is called irregular singular point.

The idea of the Frobenius Method is based upon the following form of a solution of the second-order differential equation (1.28) with regular singular point at $x = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \tag{1.30}$$

where coefficients a_n and constant α are unknown and should be obtained in such a way that function (1.30) satisfies differential equation (1.28).

We represent (1.28) in the form

$$x^2y'' + xP(x)y' + Q(x)y = 0, \tag{1.31}$$

and substitute (1.30) and (1.29) into (1.31):

$$\sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)x^{n+\alpha} + \Big\{\sum_{n=0}^{\infty} p_n x^n\Big\}\Big\{\sum_{n=0}^{\infty} a_n(n+\alpha)x^{n+\alpha}\Big\} + \Big\{\sum_{n=0}^{\infty} q_n x^n\Big\}\Big\{\sum_{n=0}^{\infty} a_n x^{n+\alpha}\Big\} = 0$$

It is seen that the left-hand side in the latter equation contains powers x^α, $x^{\alpha+1}$, $x^{\alpha+2}$ and so on. We have

$$a_0[\alpha(\alpha-1) + p_0\alpha + q_0]x^\alpha + \{a_1[\alpha(\alpha+1) + p_0(\alpha+1) + q_0] + a_0[\alpha p_1 + q_1]\}x^{\alpha+1} + ... = 0 \tag{1.32}$$

Divide both sides of the equation by x^α. The resulting polynomial on the left-hand side is identically zero if all its coefficients are zero (see Identity Principle). Therefore,

$$a_0[\alpha(\alpha-1)+p_0\alpha+q_0]=0, \tag{1.33}$$

$$a_1[\alpha(\alpha+1)+p_0(\alpha+1)+q_0]+a_0[\alpha p_1+q_1]=0, ... \tag{1.34}$$

If $a_0=0$ in (1.33), then $a_1=0$ as it follows from (1.34). Analysis of the higher order terms in (1.32) reveals that $a_n=0$, $n\geq 1$ in this case and we arrive at the trivial solution $y(x)=0$, which is of no interest. Therefore, $a_0\neq 0$ and (1.33) provides the equation with respect to α

$$\alpha(\alpha-1)+p_0\alpha+q_0=0 \tag{1.35}$$

Equation (1.35) is known as indicial equation.

THEOREM 11 If $x=0$ is a regular singular point of the differential equation (1.28) and α_1, α_2 are real solutions of the quadratic equation (1.35) such that $\alpha_1\geq\alpha_2$, then

(a) there exists a solution $y_1(x)$ of the form (1.30) with $\alpha=\alpha_1$;

(b) if $\alpha_1-\alpha_2$ is not an integer, then there exists a second solution $y_2(x)$ of the form (1.30) with $\alpha=\alpha_2$;

(c) if $\alpha_1-\alpha_2$ is integer, then there exists a second solution of the form

$$y_2(x)=Cy_1(x)\log x+\sum_{n=0}^{\infty}b_n x^{n+\alpha_2}.$$

Solutions $y_1(x)$ and $y_2(x)$ are linearly independent.

(d) if $\alpha_1=\alpha_2$, then equation (1.28) has two linearly independent solutions

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+\alpha_1},$$

$$y_2(x)=y_1(x)\log x+\sum_{n=0}^{\infty}b_n x^{n+\alpha_1+1}.$$

Bessel's differential equation

Solutions of the second-order linear differential equation

$$x^2y''+xy'+(x^2-\nu^2)y=0, \tag{1.36}$$

are known as cylinder functions or Bessel functions. Here ν is a positive parameter and $-\infty<x<\infty$. Equation (1.36) is known as Bessel's equation of order ν. In many applications, one considers a special case where the parameter ν is positive integer or zero. This case is much simpler than the case of arbitrary ν. It will serve here to introduce the general theory of Bessel functions. In the following, $\nu=m$, where $m=0,1,2,...$

Divide (1.36) by x^2 and present the Bessel's equation in standard form (1.28)

$$y'' + \frac{1}{x}y' + \frac{x^2 - m^2}{x^2}y = 0, \tag{1.37}$$

Comparing (1.37) and (1.28), we conclude that $P(x) = 1$ and $Q(x) = x^2 - m^2$.
Equations (1.29) gives

$$p_0 = 1, \quad p_n = 0 \ \ (n \geq 1),$$

$$q_0 = -m^2, \ \ q_1 = 0, \ \ q_2 = 1, \qquad q_n = 0 \ \ (n \geq 3).$$

The indicial equation (1.35) has the form

$$\alpha(\alpha - 1) + 1 \cdot \alpha - m^2 = 0.$$

Both solutions $\alpha_1 = m$ and $\alpha_2 = -m$ of this equation are real. Note that $\alpha_1 - \alpha_2 = 2m$ is integer and equals to zero if $m = 0$.

Theorem 11 (c, d) gives that two linearly independent solutions of the Bessel's equation (1.36) can be represented in the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad y_2(x) = C y_1(x) \log x + \sum_{n=0}^{\infty} b_n x^{n-m} \quad \text{if} \quad m \geq 1 \tag{1.38}$$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n}, \quad y_2(x) = y_1(x) \log x + \sum_{n=0}^{\infty} b_n x^{n+1} \quad \text{if} \quad m = 0. \tag{1.39}$$

Lecture 9

Consider $\nu = m$, $m \geq 0$, and a solution of the Bessel's equation

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m}. \tag{1.40}$$

Substitute (1.40) into (1.36). By algebra

$$\sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m} + \sum_{n=0}^{\infty} a_n (n+m) x^{n+m} + (x^2 - m^2) \sum_{n=0}^{\infty} a_n x^{n+m} = 0,$$

$$\sum_{n=0}^{\infty} [(n+m)(n+m-1) + (n+m) - m^2] a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+2} = 0,$$

$$\sum_{n=0}^{\infty} n(n+2m) a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+2} = 0,$$

$$a_0 \cdot 0 \cdot (0+2m)x^m + a_1 \cdot 1 \cdot (1+2m)x^{m+1} + \sum_{n=2}^{\infty} n(n+2m)a_n x^{n+m} + \sum_{n=2}^{\infty} a_{n-2}x^{n+m} = 0.$$

Identity principle gives

$$a_1 = 0, \qquad n(n+2m)a_n + a_{n-2} = 0 \ \ (n \geq 2).$$

Replace $n-2$ by k, then

$$a_{k+2} = -\frac{a_k}{(k+2)(k+2+2m)} \quad (k \geq 0). \tag{1.41}$$

Recurrence relation (1.41) and equality $a_1 = 0$ provide that $a_3 = 0$, $a_5 = 0$ and $a_{2n+1} = 0$. If $a_0 = 0$, then the solution is zero.

If $a_0 \neq 0$, then (1.41) gives

$$a_2 = -\frac{a_0}{(0+2)(0+2+2m)} = -\frac{a_0}{2^2 \cdot 1 \cdot (m+1)},$$

$$a_4 = \frac{a_0}{2^4 \cdot 1 \cdot 2 \cdot (m+1)(m+2)},$$

$$a_6 = -\frac{a_0}{2^6 \cdot 1 \cdot 2 \cdot 3 \cdot (m+1)(m+2)(m+3)}, ...$$

Substitute the obtained coefficients into (1.41):

$$y_1(x) = a_0 x^m - \frac{a_0}{2^2 \cdot 1 \cdot (m+1)} x^{2+m} + \frac{a_0}{2^4 \cdot 1 \cdot 2 \cdot (m+1)(m+2)} x^{4+m}$$
$$-\frac{a_0}{2^6 \cdot 1 \cdot 2 \cdot 3 \cdot (m+1)(m+2)(m+3)} x^{6+m} = ...$$

and

$$= a_0 x^m \Big\{ 1 - \frac{1}{1 \cdot (m+1)} \Big(\frac{x}{2}\Big)^2 + \frac{1}{1 \cdot 2 \cdot (m+1)(m+2)} \Big(\frac{x}{2}\Big)^4 - \frac{1}{1 \cdot 2 \cdot 3 \cdot (m+1)(m+2)(m+3)} \Big(\frac{x}{2}\Big)^6 + ... \Big\}$$

The solution $y_1(x)$ with

$$a_0 = \frac{1}{2^m m!}$$

is known as the Bessel function of the first kind of order m $J_m(x)$

$$J_m(x) = \Big(\frac{x}{2}\Big)^m \Big\{ \frac{1}{m!} - \frac{1}{1 \cdot (m+1)!} \Big(\frac{x}{2}\Big)^2 + \frac{1}{1 \cdot 2 \cdot (m+2)!} \Big(\frac{x}{2}\Big)^4 - \frac{1}{1 \cdot 2 \cdot 3 \cdot (m+3)!} \Big(\frac{x}{2}\Big)^6 + ... \Big\}$$

$$= \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}. \qquad (1.42)$$

In particular,

$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + ...,$$

$$J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^6 + ...\right\}.$$

The radius of convergence R of the power series (1.42) is calculated using (1.8) with

$$A_k = \frac{(-1)^k}{k!(k+m)!}$$

$$\frac{1}{R} = \lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{(k+1)!(k+1+m)!}\frac{k!(k+m)!}{(-1)^k}\right| = \lim_{k\to\infty}\frac{1}{(k+1)(k+1+m)} = 0.$$

Therefore, $R = \infty$ and the series (1.42) converges for any x from $-\infty$ to ∞.

Bessel functions are shown in Figure 4. Bessel function $J_n(x)$ has an infinite number of zeros x_{nk}, $J_n(x_{nk}) = 0$, $k \geq 1$. All the zeros of $J_n(x)$ are simple, except the point $x = 0$, which is a zero of order n if $n > 0$.

Second solution of the Bessel's equation (1.36) is calculated in the form (1.38) [or (1.39) if $n = 0$]. These solutions are singular at $x = 0$. Bessel functions of the first kind are regular solutions of the Bessel's equation (1.36).

THEOREM 12 The Bessel functions of higher order can be expressed in terms of the two functions $J_0(x)$ and $J_1(x)$.

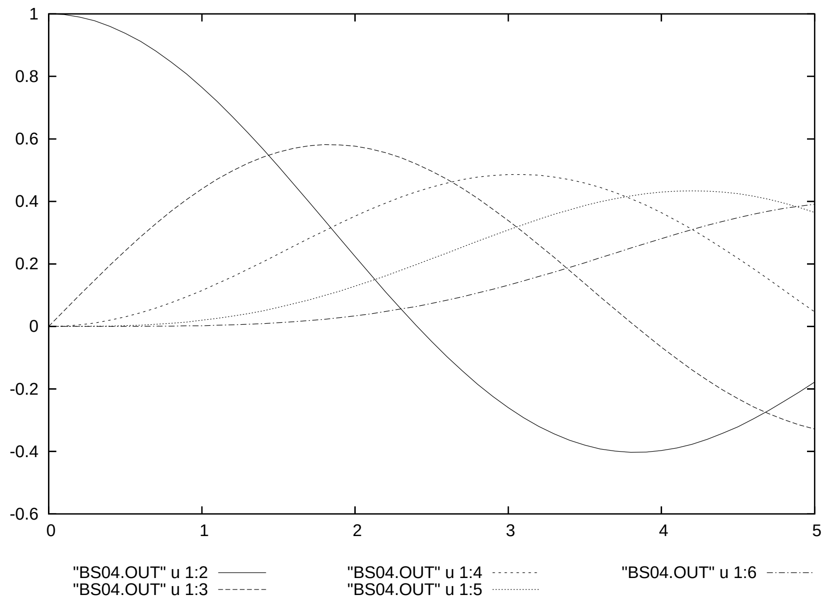
PROOF Multiply (1.42) by x^n and differentiate the result

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n J_n(x)] = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^n \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{k=0}^{\infty}\frac{(-1)^k\, 2^n}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+2n}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k(2k+2n)\, 2^n}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+2n-1}\cdot\frac{1}{2} = \sum_{k=0}^{\infty}\frac{(-1)^k(k+n)\, 2^n}{k!(k+n)!}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^{2k+n-1}$$

$$= x^n \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n-1)!}\left(\frac{x}{2}\right)^{2k+n-1} = x^n J_{n-1}(x).$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[x^n J_n(x)] = x^n J_{n-1}(x)} \qquad (n \geq 1) \qquad (1.43)$$



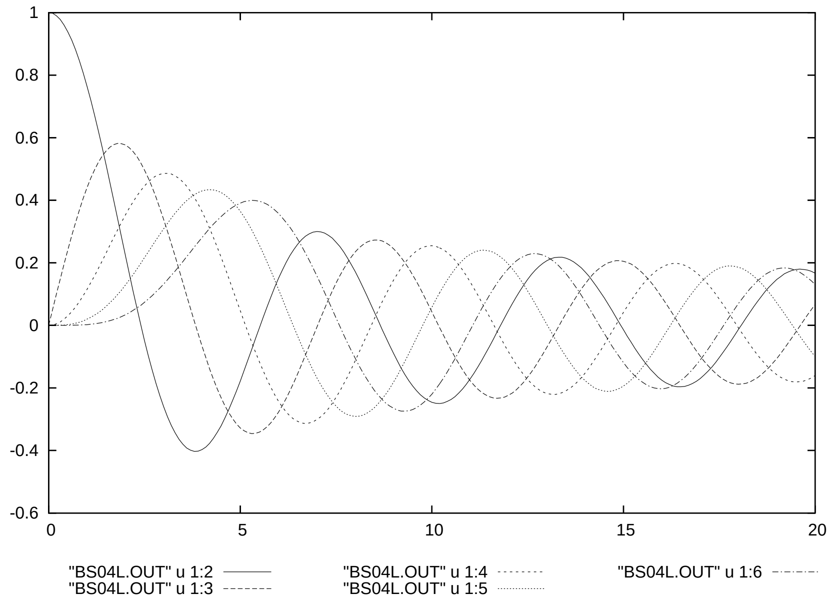


Figure 4. Bessel functions $J_n(x)$, $n = 0, 1, 2, 3, 4$, for $0 < x < 5$ and $0 < x < 20$.

Multiply (1.42) by x^{-n} and differentiate the result to obtain

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)} \qquad (n \geq 0) \tag{1.44}$$

Perform differentiation in (1.43) and (1.44), multiply the results by x^{-n} and x^n, respectively:

$$\frac{n}{x}J_n(x) + J_n'(x) = J_{n-1}(x), \qquad -\frac{n}{x}J_n(x) + J_n'(x) = -J_{n+1}(x).$$

Combine these two equations and obtain the following recurrence relations satisfied by the Bessel functions:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \qquad (n \geq 1), \tag{1.45}$$

$$J_n'(x) = -\frac{1}{2}[J_{n+1}(x) - J_{n-1}(x)] \qquad (n \geq 1). \tag{1.46}$$

In addition,

$$J_0'(x) = -J_1(x),$$

which follows from (1.44) with $n = 0$.

Relation (1.45) makes it possible to express Bessel functions of higher order in terms of the two functions $J_0(x)$ and $J_1(x)$. For example,

$n = 1: \quad J_2(x) = (2/x)J_1(x) - J_0(x)$

$n = 2: \quad J_3(x) = (4/x)J_2(x) - J_1(x) = (8/x^2 - 1)J_1(x) - (4/x)J_0(x).$

Lecture 10

REMARK

Recurrence relations (1.45) and (1.46) are valid for Bessel functions of arbitrary real order ν (not necessary integer!).

THEOREM 13 The Bessel functions of order $n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, ...$ can be expressed in terms of elementary functions.

PROOF

Consider the Bessel's equation (1.36) for $\nu = \pm\frac{1}{2}$

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0. \quad (1.47)$$

The solution of this equation is sought in the form

$$y(x) = x^{-\frac{1}{2}}v(x), \quad (1.48)$$

where $v(x)$ is a new unknown function. Substitute (1.48) into (1.47) and check that the equation becomes

$$x^{\frac{3}{2}}[v'' + v] = 0.$$

Functions $v_1(x) = \sin x$ and $v_2(x) = \cos x$ are linearly independent solutions of the latter equation. Correspondingly, $x^{-\frac{1}{2}}\sin x$ and $x^{-\frac{1}{2}}\cos x$ are linearly independent solutions of the Bessel equation (1.47). In standard notations

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x, \qquad J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos x. \quad (1.49)$$

Take $n = \frac{1}{2}$ and $n = -\frac{1}{2}$ in (1.45), use (1.49) to derive

$$J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\frac{\sin x}{x} - \cos x\right], \qquad J_{-\frac{3}{2}}(x) = -\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\frac{\cos x}{x} + \sin x\right].$$

Recurrence relation (1.45) makes it possible to express Bessel functions of order $n+\frac{1}{2}$, $n = 0, \pm1, \pm2, ...$ in terms of trigonometric and power functions. Note that the Bessel functions $J_{n+\frac{1}{2}}(x)$ are defined only for $x \geq 0$ and are singular at $x = 0$ if $n < 0$.

Zeros $x_{n+\frac{1}{2},k}$ of the Bessel functions of order $n+\frac{1}{2}$ are easier to compute than those of Bessel functions of arbitrary order. In particular, $x_{\frac{1}{2},k} = (k-1)\pi$, $k = 1, 2, 3, ...$

Expansions in series of Bessel functions

Consider $J_\nu(x)$ Bessel function of the first kind of real order ν, $\nu \geq -\frac{1}{2}$, and the positive roots $x_{\nu,1} < x_{\nu,2} < x_{\nu,3} < ... < x_{\nu,k} < ...$ of the equation $J_\nu(x) = 0$.

THEOREM 14 The system of functions $r^{\frac{1}{2}}J_\nu(x_{\nu,k}r/a)$, $k \geq 1$, is orthogonal on the interval $0 < r < a$.

PROOF The Bessel function $J_\nu(x)$ is the regular solution of the equation (1.36)

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \quad (a)$$

Consider new function $u_\alpha(r) = J_\nu(\alpha r)$. This function satisfies equation

$$u_\alpha'' + \frac{1}{r}u_\alpha' + \left(\alpha^2 - \frac{\nu^2}{r^2}\right)u_\alpha = 0. \quad (b)$$

Indeed, the left-hand side of (b) can be transformed as

$$u_\alpha'' + \frac{1}{r}u_\alpha' + \left(\alpha^2 - \frac{\nu^2}{r^2}\right)u_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big[J_\nu(\alpha r)\Big] + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big[J_\nu(\alpha r)\Big] + \left(\alpha^2 - \frac{\nu^2}{r^2}\right)J_\nu(\alpha r)$$

$$= \alpha^2 J_\nu''(\alpha r) + \frac{\alpha}{r} J_\nu'(\alpha r) + \Big(\alpha^2 - \frac{\nu^2}{r^2}\Big) J_\nu(\alpha r) = \frac{1}{r^2}\Big\{\alpha^2 r^2 J_\nu''(\alpha r) + \alpha r J_\nu'(\alpha r) + \Big(\alpha^2 r^2 - \nu^2\Big) J_\nu(\alpha r)\Big\}.$$

The result is equal to zero, which follows from (a).

Correspondingly, the function $u_\beta(r) = J_\nu(\beta r)$ satisfies the equation

$$u_\beta'' + \frac{1}{r} u_\beta' + \Big(\beta^2 - \frac{\nu^2}{r^2}\Big) u_\beta = 0. \tag{c}$$

Multiply equation (c) by $r u_\alpha(r)$ and subtract the result from equation (b) multiplied by $r u_\beta(r)$:

$$(b) \times r u_\beta(r) - (c) \times r u_\alpha(r) = r u_\alpha'' u_\beta + u_\alpha' u_\beta + \Big(\alpha^2 - \frac{\nu^2}{r^2}\Big) r u_\alpha u_\beta - r u_\beta'' u_\alpha - u_\beta' u_\alpha - \Big(\beta^2 - \frac{\nu^2}{r^2}\Big) r u_\alpha u_\beta$$

$$= (\alpha^2 - \beta^2) r u_\alpha u_\beta - \frac{\mathrm{d}}{\mathrm{d}r}\Big\{r(u_\alpha u_\beta' - u_\alpha' u_\beta)\Big\} = 0.$$

Integrate the latter equation with respect to r from 0 to a and take into account that $u_\alpha(r)$, $u_\beta(r)$ and their first derivatives are regular at $r = 0$. Obtain

$$(\alpha^2 - \beta^2) \int_0^a r u_\alpha(r) u_\beta(r) \mathrm{d}r = a[u_\alpha(a) u_\beta'(a) - u_\alpha'(a) u_\beta(a)].$$

Here $u_\alpha(r) = J_\nu(\alpha r)$ and $u_\beta(r) = J_\nu(\beta r)$:

$$(\alpha^2 - \beta^2) \int_0^a r J_\nu(\alpha r) J_\nu(\beta r) \mathrm{d}r = a[J_\nu(\alpha a) \beta J_\nu'(\beta a) - \alpha J_\nu'(\alpha a) J_\nu(\beta a)]. \tag{d}$$

Setting $\alpha = x_{\nu,k}/a$, $\beta = x_{\nu,n}/a$, we find that the RHS in (d) is zero, which gives

$$\boxed{\int_0^a r J_\nu(x_{\nu,k}\frac{r}{a}) J_\nu(x_{\nu,n}\frac{r}{a}) \mathrm{d}r = 0 \qquad (k \neq n).} \tag{e}$$

The latter equality implies that the system of functions $r^{\frac{1}{2}} J_\nu(x_{\nu,k} r/a)$, $k \geq 1$, is orthogonal on the interval $0 < r < a$.

<u>In order to calculate the integral (e) for $k = n$</u>, divide (d) by $(\alpha^2 - \beta^2)$ and take the limit of the result as $\beta \to \alpha$. Use L'Hospital's rule as

$$\int_0^a r J_\nu^2(\alpha r) \mathrm{d}r = \lim_{\beta \to \alpha} \Big\{ \frac{a[J_\nu(\alpha a) \beta J_\nu'(\beta a) - \alpha J_\nu'(\alpha a) J_\nu(\beta a)]}{\alpha^2 - \beta^2} \Big\}$$

$$= a \lim_{\beta \to \alpha} \Big\{ \frac{\frac{\partial}{\partial \beta}[J_\nu(\alpha a) \beta J_\nu'(\beta a) - \alpha J_\nu'(\alpha a) J_\nu(\beta a)]}{\frac{\partial}{\partial \beta}[\alpha^2 - \beta^2]} \Big\} = \frac{a^2}{2}\Big[J_\nu'^2(\alpha a) + \Big(1 - \frac{\nu^2}{\alpha^2 a^2}\Big) J_\nu^2(\alpha a)\Big].$$

Setting $\alpha = x_{\nu,n}/a$, we find

$$\int_0^a r J_\nu^2(x_{\nu,n}\frac{r}{a}) \mathrm{d}r = \frac{a^2}{2} J_\nu'^2(x_{\nu,n}). \tag{f}$$

By using (1.45) and (1.46) with $x = x_{\nu,n}$, we have $J'_\nu(x_{\nu,n}) = -J_{\nu+1}(x_{\nu,n})$, which makes it possible to present (f) in the final form

$$\boxed{\int_0^a rJ_\nu^2(x_{\nu,n}\frac{r}{a})\mathrm{d}r = \frac{a^2}{2}J_{\nu+1}^2(x_{\nu,n})}. \tag{g}$$