Lecture 19

We solved the boundary value problem [*Dirichlet* problem (3.27) – (3.29)] for Laplace's equation in special case where the solution $\theta(x, y)$ is zero along three sides of the rectangular domain and is prescribed, $\theta(x, b) = f(x)$, on the fourth side, $0 < x < L$, $y = b$.

In general case, the solution of a Dirichlet problem for Laplace's equation can be decomposed into sum of four solutions (see next page), each of them is obtained by the method presented in Lecture 18.

Newmann problem for a rectangle [with prescribed normal derivative on the boundary of the rectangle] is also solved by decomposing the solution into a sum of four solutions of the corresponding Newmann problems with the normal derivative being non-zero only along one side of the rectangle. Each of these four problems is solved by the method of separating variables. However, for the Newmann problems the half-range sin-Fourier series have to be substituted by the half-range cos-Fourier series.

Laplacian in polar coordinates [EK 11.9]

If we need to solve a boundary value problem for a PDE in a given region, it is a general principle to use coordinates with respect to which the boundary of the region is given by simple formulae. Boundary value problems in rectangular domains have the simplest forms in the Cartesian coordinates. Boundary value problems in circular domains (circle, half-circle, ring, . .) are studied usually in the polar coordinates r, θ such that

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{3.43}$$

In polar coordinates, for example, the boundary of a circular membrane is represented by the equation $r = const.$

In order to formulate boundary value problems in circular regions, We need to represent the Laplacian

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{3.44}$$

in polar coordinates (3.43). This implies that we consider the function

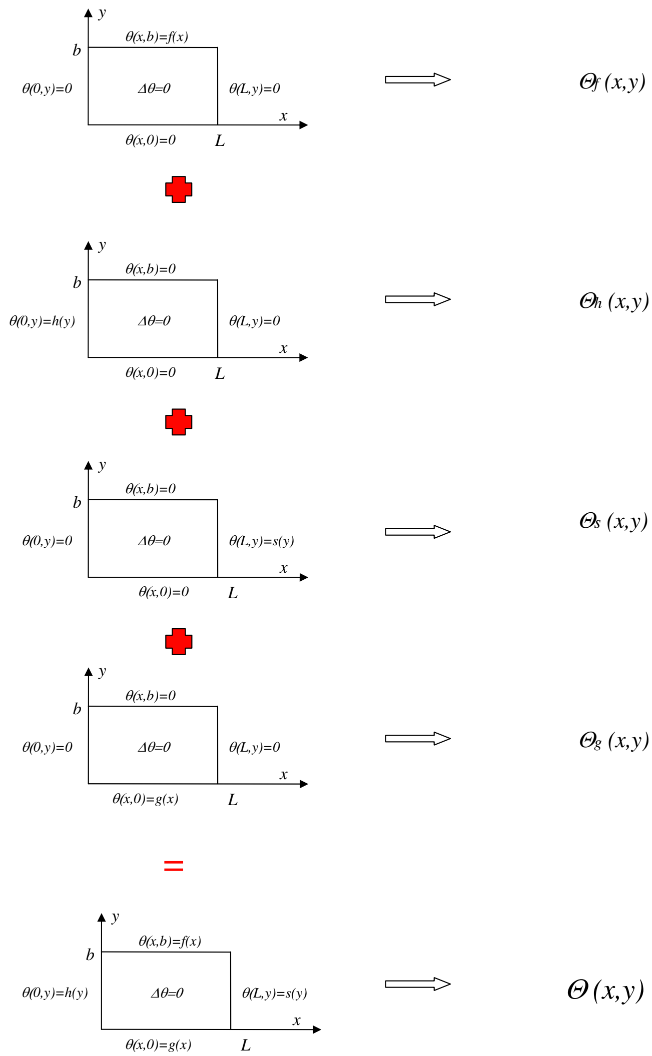
$$u(x, y) = u(r\cos\theta, r\sin\theta) =: U(r, \theta), \tag{3.45}$$

differentiate the equality (3.45) with respect to r and θ and express the second derivatives u_{xx} and u_{yy} with the help of the corresponding derivatives of the function U. For example,

$$U_r = u_x \cdot \cos\theta + u_y \cdot \sin\theta, \quad U_\theta = -u_x \cdot r\sin\theta + u_y \cdot r\cos\theta. \tag{3.46}$$

Differentiating (3.46) wrt r and θ, we can find u_{xx} and u_{yy} as a combination of the derivatives of the function $U(r, \theta)$. Substituting these relations into (3.44), we find

$$\nabla^2 u = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} \tag{3.47}$$



y
θ(x,b)=f(x)
b
θ(0,y)=0
Δθ=0
θ(L,y)=0
x
θ(x,0)=0
L
Θf (x,y)
y
θ(x,b)=0
b
θ(0,y)=h(y)
Δθ=0
θ(L,y)=0
x
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Θh (x,y)
y
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Δθ=0
θ(L,y)=s(y)
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Θs (x,y)
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θ(0,y)=0
Δθ=0
θ(L,y)=0
x
θ(x,0)=g(x)
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Θg (x,y)
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y
θ(x,b)=f(x)
b
θ(0,y)=h(y)
Δθ=0
θ(L,y)=s(y)
x
θ(x,0)=g(x)
L
Θ (x,y)

It is standard ("for the sake of simplicity") to denote $U(r,\theta)$ by the same letter as the corresponding function of x, y. Then the Laplacian in polar coordinates reads

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{3.48}$$

It can be shown (see Exercises 3) that $u_n(r,\theta) = r^n \cos(n\theta)$ and $u_n(r,\theta) = r^n \sin(n\theta)$, $n = 0, 1, 2, ...$, are solutions of Laplace's equation in polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{3.49}$$

By the method of separating variables

$$u(r,\theta) = R(r)\Theta(\theta)$$

and using the condition $\Theta(-\pi) = \Theta(\pi)$, we find

$$\Theta'' + n^2\Theta = 0$$
$$r^2R'' + rR' - n^2R = 0$$

With the solutions

$$\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$$
$$R_n(r) = c_n r^n + d_n r^{-n} \quad (n \geq 1), \quad R_0(r) = c_0 + d_0 \log(r).$$

Note that terms with d_n tends to infinity as $r \to zero$.

The regular solution of Laplace's equation (3.49) inside the circle $r < 1$ with prescribed $u(1,\theta) = f(\theta)$ is sought in the form

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\theta) + b_n \sin(n\theta)]r^n \tag{3.50}$$

Substitute this function into the boundary condition $u(1,\theta) = f(\theta)$:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

It is seen that a_n, b_n and a_0 are coefficients in Fourier series of the function $f(\theta)$. Once these coefficients have been calculated, the solution of the problem is given by (3.50).

Vibrations of circular membrane [EK 11.10]

Consider vibrations of the circular membrane of radius R in polar coordinates r, θ. The deflection $u(r,\theta,t)$ of the membrane is governed by the 2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \quad (r < R) \tag{3.51}$$

[Compare (3.51) with the 1D wave equation (3.1) derived for vibrating string in lecture 16]

We consider vibrations which are independent of the angular coordinate θ (radially symmetric vibrations). This is, we are searching for solutions $u(r,t)$ of equation (3.51). Such deflections satisfy the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) \quad (r < R). \tag{3.52}$$

The membrane is fixed along its boundary $r = R$. This provides the boundary condition

$$u(R, t) = 0 \quad (t > 0) \tag{3.53}$$

The initial conditions for equation (3.52) are

$$u(r, 0) = f(r), \quad \frac{\partial u}{\partial t}(r, 0) = g(r). \tag{3.54}$$

We shall determine the solution of the problem (3.52) – (3.54) by the method of separating variables.

Lecture 20

Step 1

We are searching for functions in the form

$$u(r, t) = F(r)G(t) \tag{a}$$

which satisfy the wave equation (3.52).
By inserting (a) into (3.52) we have

$$F(r)G''(t) = c^2[F''(r) + F'(r)/r]G(t). \tag{b}$$

Dividing (b) by $c^2F(r)G(t)$

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(r) + F'(r)/r}{F(r)}. \tag{c}$$

In (c), the LHS depends only on t and the RHS only on r. Therefore, both sides must be equal to a constant $-k^2$, say. The boundary condition (3.53) cannot be satisfied by solution (a) if the constant is positive or zero. This procedure gives us two ODEs

$$F'' + F'(r)/r + k^2F = 0, \tag{d}$$

$$G'' + c^2k^2G = 0. \tag{e}$$

Step 2

We shall now determine solutions of equations (d) and (e) such that the function (a) satisfies the boundary condition (3.53). This condition should be satisfied for any $t > 0$. This is possible if and only if

$$F(R) = 0. \tag{f}$$

This implies that we should find solutions of H-LODE2 (d) subject to homogeneous boundary conditions (f). Note that $F = 0$ is a solution of this homogeneous BVP. The question is *Are there non-trivial solutions and for which values of k these solutions can be obtained?*

We introduce new independent variable $s = kr$ in (d) and calculate the derivatives

$$\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{\mathrm{d}F}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\mathrm{d}F}{\mathrm{d}s}k, \qquad \frac{\mathrm{d}^2F}{\mathrm{d}r^2} = \frac{\mathrm{d}^2F}{\mathrm{d}s^2}k^2. \tag{g}$$

Substitute (g) and $s = kr$ into (d):

$$\frac{\mathrm{d}^2F}{\mathrm{d}s^2}k^2 + \frac{k}{kr}\frac{\mathrm{d}F}{\mathrm{d}s}k + k^2F(r) = 0$$

and

$$\frac{\mathrm{d}^2F}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}F}{\mathrm{d}s} + F(r) = 0. \tag{h}$$

Equation (h) is Bessel's equation [see (1.36) from Lecture 8] with $\nu = 0$. By Frobenius method we calculated the regular solution of that equation $J_0(r)$ and showed that the second linearly independent solution has log-singularity at $r = 0$. Therefore, the regular solution of (d) is given as

$$F(r) = J_0(s) = J_0(kr), \tag{i}$$

where the constant k is still undetermined.

Substitute (i) into the boundary condition (f):

$$F(R) = J_0(kR) = 0$$

We can see that k should be such that the product kR is a zero of the Bessel function $J_0(r)$. There are infinitely many real roots of this function. We denoted these roots as $x_{0,m}$ in Lecture 10. Here $0 < x_{0,1} < x_{0,2} < x_{0,3} < ...$ and $J_0[x_{0,m}] = 0$.

Therefore,

$$k_m = x_{0,m}/R$$

and (i) provides infinitely many functions

$$F_m(r) = J_0(x_{0,m}r/R), \tag{j}$$

which satisfy equation (d) and the condition (f).

Equation (e) now takes the form

$$G'' + \lambda_m^2 G = 0.$$

with the general solution

$$G_m(t) = A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t), \qquad (k)$$

where $\lambda_m = c x_{0,m}/R$ and coefficients A_m, B_m are undetermined.

Equations (a), (j) and (k) provide the functions

$$u_m(r,t) = [A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t)]\, J_0(x_{0,m} r/R) \qquad (l)$$

which satisfy the 2D wave equation (3.52) and the boundary conditions (3.53).

Step 3

The solutions (l) cannot satisfy the initial conditions (3.54) on their own. The Superposition Principle shows that the linear combination

$$u(r,t) = \sum_{m=1}^{\infty} u_m(r,t) \qquad (m)$$

also satisfies the wave equation (3.52) and the boundary condition (3.53).

Substitute (m) and (l) into (3.54):

$$\sum_{m=1}^{\infty} [A_m \cos(\lambda_m \cdot 0) + B_m \sin(\lambda_m \cdot 0)]\, J_0(x_{0,m} r/R) = \sum_{m=1}^{\infty} A_m J_0(x_{0,m} r/R) = f(r), \qquad (n)$$

$$\sum_{m=1}^{\infty} [-A_m \lambda_m \sin(\lambda_m \cdot 0) + B_m \lambda_m \cos(\lambda_m \cdot 0)]\, J_0(x_{0,m} r/R) = \sum_{m=1}^{\infty} B_m \lambda_m J_0(x_{0,m} r/R) = g(r). \qquad (o)$$

We can see that the coefficients A_m, B_m can be obtained as the coefficients in expansions of the given functions $f(r)$ and $g(r)$ in terms of the Bessel functions.

These series are known as Fourier-Bessel series [see lecture 11, equations (1.50) for the series and equation (1.51) for the coefficients)].

$$\boxed{f(r) = \sum_{n=1}^{\infty} c_n J_\nu(x_{\nu,n} \frac{r}{a})} \qquad (1.50)$$

$$\boxed{c_n = \frac{2}{a^2\, J_{\nu+1}^2(x_{\nu,n})} \int_0^a r f(r) J_\nu(x_{\nu,n} \frac{r}{a}) \mathrm{d}r \qquad (n = 1, 2, ...).} \qquad (1.51)$$

In the present problem we have $a = R$ and $\nu = 0$. Equations (1.51), (n) and (o) give the formulae

$$A_m = \frac{2}{R^2\, J_1^2(x_{0,\,m})} \int_0^R r f(r) J_0(x_{0,\,m}\frac{r}{R})\mathrm{d}r \qquad (n = 1, 2, ...), \tag{p}$$

$$B_m = \frac{2}{\lambda_m R^2\, J_1^2(x_{0,\,m})} \int_0^R r g(r) J_0(x_{0,\,m}\frac{r}{R})\mathrm{d}r \qquad (n = 1, 2, ...). \tag{r}$$

Formulae (l), (m), (p) and (r) provide the solution of the problem for vibrating membrane. The integrals in (p) and (r) are calculated numerically.

EXAMPLE

Consider $R = 1$, $g(r) = 0$ and the initial shape of the membrane $f(r) = J_0(x_{0,\,1}\frac{r}{R})$.

Equation (o) gives $B_m = 0$. It follows from equation (n) that $A_1 = 1$ and $A_m = 0$ for $m \geq 2$. The solution is obtained by using (l) and (m) as

$$u(r, t) = \cos[\lambda_1 t] J_0(x_{0,\,1}\frac{r}{R}),$$

where $\lambda_1 = c x_{0,\,1}/R$ is the frequency of the vibrations.

Laplace's equation in spherical coordinates [EK 11.11,12]

Spherical coordinates r, ϕ, θ are defined as

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi, \tag{3.55}$$

where $r > 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$.

The Laplacian of a function $u(r, \phi, \theta)$ in spherical coordinates is

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right]. \tag{3.56}$$

We consider a typical boundary value problem for Laplace's equation in spherical coordinates.

Consider a solid sphere of radius R with given steady temperature on the boundary of the sphere $f(\phi)$, which is independent of the coordinate θ. We need to find steady temperature distribution $u(r, \phi)$ inside the sphere, subject to the boundary condition

$$u(R, \phi) = f(\phi). \tag{3.57}$$

The steady temperature distribution satisfies Laplace's equation which in spherical coordinates reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) = 0 \qquad (r < R). \tag{3.58}$$

Show that $u_n(r,\phi) = r^n P_n(\cos\phi)$ and $v_n(r,\phi) = r^{-(n+1)} P_n(\cos\phi)$, $n = 0, 1, 2, ...$, are solutions of Laplace's equation (3.58) [Exercise 7].

Here $P_n(x)$ are Legendre polynomials, $P_0(x) = 1$, $P_1(x) = x$, . ., which are the regular solutions of the equations

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

Solutions $v_n(r,\phi)$ are singular at $r = 0$. Linear combination of the regular (inside the sphere) solutions $u_n(r,\phi)$

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi) \tag{3.59}$$

is also a solution of (3.58). Substitute (3.59) into the boundary condition (3.57)

$$u(R,\phi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\phi) = f(\phi) \tag{3.60}$$

Introduce new variable $w = \cos\phi$, $-1 < w < 1$ and function $\tilde{f}(w)$ such that

$$\tilde{f}(\cos\phi) = f(\phi).$$

Then equation (3.60) takes the form

$$\tilde{f}(w) = \sum_{n=0}^{\infty} A_n R^n P_n(w) \quad (-1 < w < 1)$$

and represents the expansion of the function $\tilde{f}(w)$ in terms of Legendre polynomials [see Lecture 7, equation (1.23) for the series form and equation (1.24) for the coefficients as integrals of $\tilde{f}(w)$].

Formula (1.24) gives

$$A_n R^n = (n + \frac{1}{2}) \int_{-1}^{1} \tilde{f}(w) P_n(w) dw$$

Since $dw = -\sin\phi d\phi$ and the limits of integration -1 and 1 correspond to $\phi = \pi$ and $\phi = 0$, respectively, we have

$$A_n = R^{-n}(n + \frac{1}{2}) \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi d\phi \quad (n \geq 0). \tag{3.61}$$

The series (3.59) with coefficients (3.61) is the solution of the problem for steady temperature distribution inside the sphere.