Lecture 11

Consider a function $f(r)$ defined on an interval $(0, a)$.

Is it possible to present this function as a series involving Bessel functions

$$f(r) = \sum_{n=1}^{\infty} c_n J_\nu(x_{\nu,n} \frac{r}{a}) \tag{1.50}$$

and, if yes, how to find the coefficients c_n *in (1.50)?*

Coefficients c_n are obtained by multiplying (1.50) by $r\, J_\nu(x_{\nu,m} \frac{r}{a})$ and integrating the result term by term over the interval $(0, a)$. Using (e) and (g), we find

$$\int_0^a r f(r) J_\nu(x_{\nu,m} \frac{r}{a}) \mathrm{d}r = c_m \frac{a^2}{2} J_{\nu+1}^2(x_{\nu,m}).$$

and, therefore,

$$\boxed{c_n = \frac{2}{a^2\, J_{\nu+1}^2(x_{\nu,n})} \int_0^a r f(r) J_\nu(x_{\nu,n} \frac{r}{a}) \mathrm{d}r \qquad (n = 1, 2, ...).} \tag{1.51}$$

DEFINITION 17 The series (1.50) with coefficients c_n calculated from (1.51) is called Fourier-Bessel series.

THEOREM 15 If a function $f(r)$ is piecewise smooth in $(0, a)$ and if the integral

$$\int_0^a \sqrt{r} |f(r)| \mathrm{d}r$$

is finite, then the Fourier-Bessel series (1.50), with coefficients c_n given by (1.51), converges to $f(r)$ at every continuity point of $f(r)$ and to

$$\frac{1}{2}[f(r-0) + f(r+0)],$$

if r is a discontinuity point of $f(r)$.

DEFINITION 18 The series (1.50) where $x_{\nu,n}$ are positive roots of the equation

$$x J'_\nu(x) + B J_\nu(x) = 0 \tag{1.52}$$

are called Dini series.

THEOREM 16 The system of functions $r^{\frac{1}{2}} J_\nu(x_{\nu,k} r/a)$, $k \geq 1$, where $x_{\nu,n}$ are positive roots of the equation (1.52), is orthogonal on the interval $0 < r < a$.

SKETCH OF PROOF The proof is similar to that in Theorem 14. Setting $\alpha = x_{\nu,k}/a$, $\beta = x_{\nu,n}/a$ in (d), where $x_{\nu,n}$ are positive roots of the equation (1.52), show that the RHS in (d) is equal to zero if $n \neq k$. This implies that the system of functions is orthogonal.

SUMMARY

1. We studied LODE2 in standard form

$$y'' + p(x)y' + q(x)y = r(x). \tag{0.1}$$

2. A general solution $y_{GNH}(x)$ of the nonhomogeneous LODE (0.1) is given by

$$y_{GNH}(x) = y_{GH}(x) + y_{PNH}(x), \tag{0.8}$$

3. A general solution $y_{GH}(x)$ of the associated homogeneous equation is given by

$$y_{GH}(x) = c_1 y_1(x) + c_2 y_2(x) \tag{0.5}$$

where $y = y_1(x)$ and $y = y_2(x)$ are linearly independent solutions of the associated homogeneous equation.

4. Two solutions $y = y_1(x)$ and $y = y_2(x)$ are linearly independent (not proportional) on I if and only if the Wronskian $W(y_1, y_2)(x_0) \neq 0$ at a single point of I.

5. A particular solution $y_{PNH}(x)$ can be obtained by the method of variation of parameters

$$y_{PNH}(x) = -y_1(x) \int \frac{y_2(x) r(x)}{W(x)} \mathrm{d}x + y_2(x) \int \frac{y_1(x) r(x)}{W(x)} \mathrm{d}x. \tag{0.12}$$

if two linearly independent solutions $y = y_1(x)$ and $y = y_2(x)$ are given.

6. If a solution $y_1(x)$ of the H-LODE2 is known, then a second linearly independent solution $y_2(x)$ can be found as

$$y_2(x) = y_1(x) \int y_1^{-2}(x) \exp\Big(-\int p(x)\mathrm{d}x\Big)\mathrm{d}x. \tag{0.14}$$

7. We studied how to solve the IVP

$$y'' + p(x)y' + q(x)y = r(x) \quad (x \in I), \tag{0.16}$$

$$y(x_0) = K_0, \quad y'(x_0) = K_1 \quad (x_0 \in I), \tag{0.17}$$

8. We showed that the solution of the BVP

$$\begin{gathered} y'' + p(x)y' + q(x)y = r(x) \quad (a \leq x \leq b), \\ \alpha y(a) + \beta y'(a) = K_0, \qquad \lambda y(b) + \mu y'(b) = K_1 \end{gathered} \tag{0.20}$$

can be not unique or does not exist.

9. If both $p(x)$ and $q(x)$ in (0.1) have Taylor series, which converge on the interval $I = (-R, R)$, $R > 0$, and $r(x) = 0$, then solutions $y = y_1(x)$ and $y = y_2(x)$ can be obtained by Power Series Method in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{1.12}$$

10. We studied Legendre equation

$$(1 - x^2)y'' - 2xy' + m(m+1)y = 0 \tag{1.10}$$

For integer m a regular solution of (1.10) which is equal to unity at $x = 1$ is known as the Legendre polynomial, $P_m(x)$, of degree m.

11. We checked Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2 - 1)^n]. \tag{1.18}$$

12. We showed recurrence relations

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad (n = 1, 2, 3, ...) \tag{1.19}$$

$$P'_{n+1}(x) = P'_{n-1}(x) + (2n+1)P_n(x). \tag{1.20}$$

13. We defined orthogonal systems of functions $\varphi_n(x)$, $(n = 0, 1, 2, ...)$

$$\int_a^b \varphi_n(x)\varphi_m(x)\mathrm{d}x = 0 \quad (n \neq m).$$

and orthonormal systems

$$\int_a^b \varphi_n^2(x)\mathrm{d}x = 1.$$

14. We proved that the Legendre polynomials $P_n(x)$, $n \geq 0$, are orthogonal on the interval $(-1, 1)$ and

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2}{2n+1} \quad (n = 0, 1, 2, ...) \tag{1.22}$$

15. If a function $f(x)$ is piecewise smooth in $(-1, 1)$ and if the integral

$$\int_{-1}^{1} f^2(x)\mathrm{d}x$$

is finite, then the function can be expanded in series of Legendre polynomials

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \tag{1.23}$$

16. If associated homogeneous equation can be presented as

$$y'' + \frac{P(x)}{x}y' + \frac{Q(x)}{x^2}y = 0, \tag{1.28}$$

where

$$P(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad Q(x) = \sum_{n=0}^{\infty} q_n x^n \qquad (|x| < R) \tag{1.29}$$

then a solution can be found by **Frobenius Method** in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \tag{1.30}$$

where α satisfies the indicial equation

$$\alpha(\alpha - 1) + p_0\alpha + q_0 = 0 \tag{1.35}$$

17. Solutions of Bessel's differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{1.36}$$

can be found by the Frobenius Method. The regular solution of (1.36) is known as Bessel function of first kind of order m

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!}\left(\frac{x}{2}\right)^{2k+m}. \tag{1.42}$$

18. The Bessel functions of higher order can be expressed in terms of the two functions $J_0(x)$ and $J_1(x)$ [Th12].

19. We studied recurrence relations

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \qquad (n \geq 1), \tag{1.45}$$

$$J'_n(x) = -\frac{1}{2}[J_{n+1}(x) - J_{n-1}(x)] \qquad (n \geq 1), \tag{1.46}$$

$$J'_0(x) = -J_1(x).$$

20. The Bessel functions of order $n + \frac{1}{2}$, $n = 0, \pm1, \pm2, ...$ can be expressed in terms of elementary functions. In particular,

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x, \qquad J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x. \tag{1.49}$$

21. The system of functions $r^{\frac{1}{2}}J_\nu(x_{\nu,k}r/a)$, $k \geq 1$, is orthogonal on the interval $0 < r < a$ [Th14]. Here $x_{\nu,1} < x_{\nu,2} < x_{\nu,3} < ... < x_{\nu,k} < ...$ are positive roots of the equation $J_\nu(x) = 0$.

22. We considered Fourier-Bessel series

$$f(r) = \sum_{n=1}^{\infty} c_n J_\nu(x_{\nu,n}\frac{r}{a}) \tag{1.50}$$

and stated that if a function $f(r)$ is piecewise smooth in $(0,a)$ and if the integral

$$\int_0^a \sqrt{r}|f(r)|\mathrm{d}r$$

is finite, then the Fourier-Bessel series (1.50) with coefficients c_n given by

$$c_n = \frac{2}{a^2\, J_{\nu+1}^2(x_{\nu,\, n})} \int_0^a rf(r) J_\nu(x_{\nu,\, n}\frac{r}{a})\mathrm{d}r \qquad (n = 1, 2, ...). \tag{1.51}$$

converges.

23. The system of functions $r^{\frac{1}{2}} J_\nu(x_{\nu,k} r/a)$, $k \geq 1$, is orthogonal on the interval $0 < r < a$, where $x_{\nu,1} < x_{\nu,2} < x_{\nu,3} < ... < x_{\nu,k} < ...$ are positive roots of the equation

$$xJ'_\nu(x) + BJ_\nu(x) = 0. \tag{1.52}$$

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS [EK, 2.9,2.11]

We consider LODEn

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + ...p_1(x)y' + p_0(x)y = r(x) \tag{2.1}$$

and the associated homogeneous equation

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + ...p_1(x)y' + p_0(x)y = 0. \tag{2.2}$$

In standard form, $p_n(x) = 1$ in (2.1) and (2.2).Only this case is considered in this module. It is assumed that the coefficients $p_j(x)$, $0 \leq j \leq n-1$ are continuous on an open interval $I = (a, b)$.

A general solution of (2.2) is

$$y_{GH}(x) = \sum_{k=1}^{\infty} c_k y_k(x), \tag{2.3}$$

where $y_1(x)$, $y_2(x)$, . . . , $y_n(x)$ are linearly independent solutions of (2.2) and $c_1, c_2, ..., c_n$ are arbitrary real constants.

Functions $y_1(x)$, $y_2(x)$, . . . , $y_n(x)$ are linearly independent on interval $I = (a, b)$ if the equation

$$\sum_{k=1}^{\infty} b_k y_k(x) = 0 \qquad (a < x < b)$$

implies that all $b_1, b_2, ..., b_n$ are zero.

THEOREM 17 Solutions $y_1(x)$, $y_2(x)$, . . . , $y_n(x)$ of (2.2) are linearly dependent on interval $I = (a, b)$ if and only if their <u>Wronskian</u> $W(y_1, .., y_n)(x)$ is zero at some point $x_0 \in I$.

The Wronskian is defined as

$$W(y_1,..,y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & ... & y_n(x) \\ y_1'(x) & y_2'(x) & ... & y_n'(x) \\ ... & ... & ... & ... \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & ... & y_n^{(n-1)}(x) \end{vmatrix}$$

A general solution of the NH-LODEn (2.1) is given by

$$y_{GNH}(x) = y_{GH}(x) + y_{PNH}(x),$$

where $y_{PNH}(x)$ is a particular solution of (2.1).

An <u>initial value problem</u> for equation (2.1) consists of equation (2.1) and n <u>initial conditions</u>

$$y(x_0) = K_0, \quad y'(x_0) = K_1, .. y^{(n-1)}(x_0) = K_{n-1}. \tag{2.4}$$

The solution of the IVP (2.1), (2.4) is unique in I if the coefficients in (2.1) are continuous functions on I.

Lecture 12

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF ORDER n WITH CONSTANT COEFFICIENTS [EK, 2.10]

Consider equation (2.2) where the coefficients p_j are real constants and $p_n = 1$. A solution is sought in the form $y(x) = \exp(rx)$. Substitution of $y(x) = \exp(rx)$ and its derivatives into (2.2) gives the <u>characteristic equation</u>

$$Q(r) = r^n + p_{n-1}r^{n-1} + .. + p_1 r + p_0 = 0 \tag{2.5}$$

Let r be a simple root of (2.5).
If r is real, then $y(x) = e^{rx}$ is a solution of (2.2).
If $r = \alpha + i\beta$ is a complex root, then $\alpha - i\beta$ is also a root since the coefficients in (2.2) are real. These two roots generate two linearly independent solutions

$$e^{\alpha x}\cos(\beta x) \quad \text{and} \quad e^{\alpha x}\sin(\beta x).$$

Let r be a root of (2.5) with multiplicity m.
If r is real, then

$$e^{rx},\ xe^{rx}, \ldots, x^{m-1}e^{rx}$$

are m linearly independent solutions of (2.2).

If $r = \alpha + i\beta$ is a complex root, then $\alpha - i\beta$ is also a root with multiplicity m. These two roots generate $2m$ linearly independent solutions

$$e^{\alpha x}\cos(\beta x), \quad xe^{\alpha x}\cos(\beta x), \ldots, x^{m-1}e^{\alpha x}\cos(\beta x),$$
$$e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\sin(\beta x), \ldots, x^{m-1}e^{\alpha x}\sin(\beta x).$$

Therefore, to find a general solution of the homogeneous equation (2.2) with constant coefficients, we need to calculate the roots of equation (2.5).

EXAMPLE:

$$y''' - 3y'' + 3y' - y = 0$$

$$r^3 - 3r^2 + 3r - 1 = 0, \qquad r^3 - 3r^2 + 3r - 1 = (r-1)^3$$

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = 1.$$

Therefore, the characteristic equation has the root $r = 1$ with multiplicity 3. The linearly independent solutions are

$$y_1(x) = e^x, \quad y_2(x) = xe^x, \quad y_3(x) = x^2e^x$$

and a general solution reads

$$y(x) = e^x[c_1 + c_2x + c_3x^2].$$

A particular solution of the equation (2.1) with constant coefficients can be found by method of undetermined coefficients, if $r(x) = e^{ax}\cos(bx)P(x)$ or $r(x) = e^{ax}\sin(bx)P(x)$, where $P(x)$ is a polynomial function.

Step 1: Find roots of the characteristic equation (2.5) and their multiplicities.

Step 2: Generate the complex number $z = a + ib$.

If $Q(z) \neq 0$, then set $s = 0$;

If $Q(z) = 0$, then z is a root of (2.5) and s is its multiplicity.

Step 3: A particular solution of (2.1) is sought in the form

$$y_{PNH}(x) = x^s e^{ax}[T(x)\cos(bx) + R(x)\sin(bx)],$$

where $T(x)$ and $R(x)$ are two polynomial functions with $degree(T) = degree(R) = degree(P)$. If

$$P(x) = \sum_{k=0}^{m} \nu_k x^k,$$

where ν_k are given, then

$$T(x) = \sum_{k=0}^{m} \lambda_k x^k, \qquad R(x) = \sum_{k=0}^{m} \mu_k x^k,$$

where the coefficients λ_k and μ_k, $0 \leq k \leq m$, should be determined ($2m+2$ unknown coefficients in total).

Step 4: Plug $y_{PNH}(x)$ into the equation (2.1) and calculate the unknown coefficients λ_k and μ_k.

Step 5: Write down a particular solution $y_{PNH}(x)$.

EXAMPLE:

$$y''' - y' = \sin(bx)$$

Step 1: $Q(r) = r^3 - r = 0, \quad r(r^2-1) = 0 \Longrightarrow r_1 = 0,\, r_2 = 1,\, r_3 = -1$

Step 2: $z = ib \Longrightarrow Q(z) \neq 0 \Longrightarrow s = 0$

Step 3: $P(x) = 1$, $y(x) = T\cos(bx) + R\sin(bx)$, where T and R are real constants.

Step 4: $y'(x) = -Tb\sin(bx) + Rb\cos(bx)$, $y'''(x) = Tb^3\sin(bx) - Rb^3\cos(bx)$,

$y''' - y' = Tb^3\sin(bx) - Rb^3\cos(bx) + Tb\sin(bx) - Rb\cos(bx)$

$y''' - y' = T[b^3+b]\sin(bx) - R[b^3+b]\cos(bx) = \sin(bx)$

$T[b^3+b] = 1,\ R = 0.$

Step 5:

$$\boxed{y(x) = \frac{\cos(bx)}{b^3+b}}$$

EXAMPLE:

A particular solution of the equation

$$y''' - y' = A\sin(bx)$$

is

$$\boxed{y(x) = \frac{A\cos(bx)}{b^3+b}}$$

EXAMPLE:

A particular solution of the equation

$$y''' - y' = A\cos(bx)$$

is

$$\boxed{y(x) = \frac{-A\sin(bx)}{b^3 + b}}$$

EXAMPLE:

A particular solution of the equation

$$y''' - y' = \sin(x) + 2\sin(2x)$$

is

$$y(x) = \frac{1 \cdot \sin(x)}{1^3 + 1} + \frac{2 \cdot \sin(2x)}{2^3 + 2}$$

EXAMPLE:

A particular solution of the equation

$$y''' - y' = \sum_{n=1}^{N} a_n \sin(nx)$$

is

$$y(x) = \sum_{n=1}^{N} \frac{a_n}{n^3 + n} \cos(nx)$$

Lecture 13

GENERAL THEORY:

Consider equation (2.1) with constant coefficients

$$\sum_{k=0}^{n} p_k y^{(k)} = r(x) \tag{2.6}$$

on the interval $I = (-\pi, \pi)$ and assume that

$$r(x) = \sum_{m=0}^{\infty} [a_m \cos(mx) + b_m \sin(mx)], \tag{2.7}$$

where coefficients a_m, b_m are given and decay as $m \to \infty$.

Note that

$$a_m \cos(mx) + b_m \sin(mx) = \Re\{(a_m - ib_m)e^{imx}\}.$$

Denote $A_m = a_m - ib_m$ and write down (2.7) as

$$r(x) = \Re\left\{\sum_{m=0}^{\infty} A_m e^{imx}\right\}. \tag{2.8}$$

A particular solution of (2.6) with the RHS (2.8) is sought in the form

$$y(x) = \Re\left\{\sum_{m=0}^{\infty} A_m R_m e^{imx}\right\} \tag{2.9}$$

with unknown complex coefficients R_m. Calculate

$$y^{(k)}(x) = \Re\left\{\sum_{m=0}^{\infty} A_m R_m (im)^k e^{imx}\right\}. \tag{2.10}$$

Plug (2.10) and (2.8) into equation (2.6)

$$\sum_{k=0}^{n} p_k \Re\left\{\sum_{m=0}^{\infty} A_m R_m (im)^k e^{imx}\right\} = \Re\left\{\sum_{m=0}^{\infty} A_m e^{imx}\right\}$$

and change the order of summation on the left-hand side

$$\Re\left\{\sum_{m=0}^{\infty} A_m R_m \left[\sum_{k=0}^{n} p_k (im)^k\right] e^{imx}\right\} = \Re\left\{\sum_{m=0}^{\infty} A_m e^{imx}\right\}$$

or

$$\Re\left\{\sum_{m=0}^{\infty} A_m R_m Q(im) e^{imx}\right\} = \Re\left\{\sum_{m=0}^{\infty} A_m e^{imx}\right\},$$

$$\Re\left\{\sum_{m=0}^{\infty} A_m[R_m Q(im) - 1]e^{imx}\right\} = 0,$$

where the function $Q(r)$ is defined by (2.5). We assume that $Q(im) \neq 0$ for $m \geq 0$.

Therefore, $R_m = 1/Q(im)$ and a particular solution of (2.6) is

$$y(x) = \Re\left\{\sum_{m=0}^{\infty} \frac{A_m}{Q(im)} e^{imx}\right\}. \tag{2.11}$$

IMPORTANT If the RHS $r(x)$ in differential equation (2.6) can be presented in the form (2.7), then a particular solution of (2.6) is given by (2.11).

IMPORTANT The series (2.7) represents a periodic function of period 2π.

DEFINITION 19 A function $r(x)$ is said to be periodic if it is defined for all real x and if there is a positive number p such that $r(x + p) = r(x)$. The number p is called a period of $f(x)$.

IMPORTANT The solution (2.11) is a periodic solution of (2.6) for the periodic RHS $r(x)$.

Check that $p = 2\pi$ for the function defined by (2.7).

DEFINITION 20 The series (2.7) is known as Fourier series.

FOURIER SERIES [EC 10.2]

We assume that $f(x)$ is a periodic function of period 2π. The Fourier series of the function $f(x)$ is a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], \tag{2.12}$$

whose coefficients a_n and b_n are determined from $f(x)$ by Euler formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{d}x, \tag{2.13}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x, \tag{2.14}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x. \tag{2.15}$$

Derivation of (2.13-15)

(2.13)

Integrate both sides of (2.12) from $-\pi$ to π term by term

$$\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \int_{-\pi}^{\pi} a_0\mathrm{d}x + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin(nx)\mathrm{d}x\right],$$

$$\int_{-\pi}^{\pi}\cos(nx)\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi}\sin(nx)\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi} a_0\mathrm{d}x = 2\pi a_0,$$

$$\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 2\pi a_0 \Longrightarrow \text{(2.13)!}$$

(2.14)

Multiply (2.12) by $\cos(mx)$, where m is any fixed positive integer, and integrate both sides of the resulting equation from $-\pi$ to π term by term

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\mathrm{d}x = a_0\int_{-\pi}^{\pi}\cos(mx)\mathrm{d}x + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\mathrm{d}x\right]$$

By using trigonometric formulae

$$\cos A\cos B = \frac{1}{2}\left\{\cos(A+B) + \cos(A-B)\right\}, \quad \sin A\cos B = \frac{1}{2}\left\{\sin(A+B) + \sin(A-B)\right\},$$

we have

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\mathrm{d}x = a_m\int_{-\pi}^{\pi}\cos^2(mx)\mathrm{d}x.$$

Here

$$\cos^2(mx) = \frac{1}{2}\left\{1 + \cos(2mx)\right\}, \quad \int_{-\pi}^{\pi}\cos^2(mx)\mathrm{d}x = \pi + 0 \Longrightarrow \text{(2.14)}.$$

(2.15)

Multiply (2.12) by $\sin(mx)$, where m is any fixed positive integer, and integrate both sides of the resulting equation from $-\pi$ to π term by term

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\mathrm{d}x = a_0\int_{-\pi}^{\pi}\sin(mx)\mathrm{d}x+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\mathrm{d}x\right]$$

By using trigonometric formulae

$$\sin A\sin B = \frac{1}{2}\left\{\cos(A-B)-\cos(A+B)\right\}, \quad \sin A\cos B = \frac{1}{2}\left\{\sin(A+B)+\sin(A-B)\right\},$$

we have

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\mathrm{d}x = b_m\frac{1}{2}\int_{-\pi}^{\pi}\cos(mx-mx)\mathrm{d}x = b_m\pi \Longrightarrow (2.15).$$

The coefficients given by (2.13) - (2.15) are called the <u>Fourier coefficients</u> of $f(x)$.

EXAMPLE

Consider $f(x) = -k$, where $-\pi < x < 0$, and $f(x) = k$, where $0 < x < \pi$. Calculate the Fourier coefficients of $f(x)$.

This is an odd function, $f(-x) = -f(x)$, defined on the symmetric interval $-\pi < x < \pi$. Therefore, $a_n = 0$, $n \geq 0$. Formula (2.15) gives

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin(nx)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi}k\sin(nx)\mathrm{d}x$$

$$= -\frac{k}{\pi}\int_{-\pi}^{0}\sin(nx)\mathrm{d}x + \frac{k}{\pi}\int_{0}^{\pi}\sin(nx)\mathrm{d}x = -\frac{k}{\pi}\left(\frac{-\cos(nx)}{n}\right)_{-\pi}^{0} + \frac{k}{\pi}\left(\frac{-\cos(nx)}{n}\right)_{0}^{\pi}$$

$$= \frac{k}{\pi}\frac{1-\cos(n\pi)}{n} + \frac{k}{\pi}\frac{1-\cos(nx)}{n} = \frac{2k}{n\pi}[1-\cos(n\pi)] = \frac{2k}{n\pi}[1-(-1)^n],$$

$$\boxed{b_{2m} = 0, \qquad b_{2m+1} = \frac{4k}{(2m+1)\pi}}$$

The Fourier series of our function has the form

$$f(x) = \frac{4k}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n+1)x]}{2n+1}$$

This Fourier series, in particular, provides a useful formula

$$f(\frac{\pi}{2}) = k = \frac{4k}{\pi}\left(\sin(\frac{\pi}{2}) + \frac{1}{3}\sin(\frac{3\pi}{2}) + \frac{1}{5}\sin(\frac{5\pi}{2}) + \dots\right)$$

or

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}}$$

Lecture 14

THEOREM 17 If a function $f(x)$ defined in the interval $[-\pi, \pi]$ and its first derivative $f'(x)$ are continuous, satisfy the conditions

$$f(-\pi) = f(\pi), \qquad f'(-\pi) = f'(\pi) \tag{2.16}$$

and the second derivative is integrable

$$\int_{-\pi}^{\pi} |f''(x)|\mathrm{d}x \leq C, \tag{2.17}$$

then the Fourier series (2.12), with coefficients a_n and b_n given by (2.13) – (2.15), converges at every point of the interval.

SKETCH OF PROOF Integrate (2.14) and (2.15) by parts twice and use conditions (2.16). This gives

$$a_n = \frac{1}{n^2}\int_{-\pi}^{\pi} f''(x)\cos(nx)\mathrm{d}x.$$

Estimate $|a_n|$ as

$$|a_n| = \left|\frac{1}{n^2}\int_{-\pi}^{\pi} f''(x)\cos(nx)\mathrm{d}x\right| \leq \frac{1}{n^2}\int_{-\pi}^{\pi} |f''(x)\cos(nx)|\,\mathrm{d}x$$

$$\leq \frac{1}{n^2}\int_{-\pi}^{\pi} |f''(x)||\cos(nx)|\mathrm{d}x \leq \frac{1}{n^2}\int_{-\pi}^{\pi} |f''(x)|\mathrm{d}x.$$

Use (2.17) to deduce

$$|a_n| \leq \frac{C}{n^2}.$$

In a similar way find

$$|b_n| \leq \frac{C}{n^2}.$$

These inequalities give

$$|a_n \cos(nx) + b_n \sin(nx)| \le |a_n| + |b_n| \le \frac{2C}{n^2}$$

which implies that the series (2.12) converges for any $x \in [-\pi, \pi]$.

THEOREM 18 If a function $f(x)$ defined in the interval $[-\pi, \pi]$, its first and second derivatives are piecewise continuous, then the Fourier series (2.12), with coefficients a_n and b_n given by (2.13) – (2.15), converges to $f(x)$ at every continuity point of $f(x)$ and to

$$\frac{1}{2}[f(x-0) + f(x+0)],$$

if x is a discontinuity point of $f(x)$.

IMPORTANT Fourier series (2.12) is defined in unique way for a given function $f(x)$ and given interval of x.

FOURIER SERIES OF FUNCTIONS DEFINED IN INTERVAL [a, a+2L] [EK, 10.3]

Consider a function $f(x)$ defined in the interval $(a, a+2L)$, where a can be different from $-\pi$ and $2L$ can be different from 2π. The function $f(x)$ is assumed to be piecewise smooth.

Define the new function $F(v)$ as $[x = vL/\pi + a + L]$

$$F(v) = f(vL/\pi + a + L), \qquad (-\pi < v < \pi). \tag{2.18}$$

The function $F(v)$ is defined on the interval $-\pi < v < \pi$ and is piecewise smooth. This function can be presented by its Fourier series as

$$F(v) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(nv) + B_n \sin(nv)], \tag{2.19}$$

where the coefficients A_n and B_n are given by <u>Euler formulas</u> (2.13) – (2.15):

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(v) \mathrm{d}v,$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(v) \cos(nv) \mathrm{d}v, \qquad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(v) \sin(nv) \mathrm{d}v. \tag{2.20}$$

Substituting $v = (x - a - L)\pi/L$ into (2.19) and using (2.18), we can write down (2.19) as

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\pi n}{L}(x - a - L) \right) + B_n \sin\left(\frac{\pi n}{L}(x - a - L) \right) \right]. \tag{2.21}$$

Here

$$\cos\left(\frac{\pi n}{L}(x-a-L)\right) = \cos\left(\frac{\pi n}{L}(x-a)\right)(-1)^n, \quad \sin\left(\frac{\pi n}{L}(x-a-L)\right) = \sin\left(\frac{\pi n}{L}(x-a)\right)(-1)^n. \tag{2.22}$$

Substituting $v = (x-a-L)\pi/L$ into (2.20) and using (2.18), (2.22) we find

$$A_0 = \frac{1}{2L}\int_a^{a+2L} f(x)\mathrm{d}x =: a_0,$$

$$A_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\left(\frac{\pi n}{L}(x-a)\right)\mathrm{d}x(-1)^n =: a_n(-1)^n,$$

$$B_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{\pi n}{L}(x-a)\right)\mathrm{d}x(-1)^n =: b_n(-1)^n \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.21), we arrive at the Fourier series for a function defined in an arbitrary interval

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{\pi n}{L}(x-a)\right) + b_n\sin\left(\frac{\pi n}{L}(x-a)\right)\right], \tag{2.24}$$

$$a_0 = \frac{1}{2L}\int_a^{a+2L} f(x)\mathrm{d}x,$$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\left(\frac{\pi n}{L}(x-a)\right)\mathrm{d}x,$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{\pi n}{L}(x-a)\right)\mathrm{d}x. \tag{2.25}$$

EXAMPLE

Consider a piecewise smooth function $f(x)$ defined in the interval $(0, 2\pi)$. Equations (2.24) and (2.25) with $a = 0$ and $L = \pi$ provide the Fourier series for this function and the coefficients of the series as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right],$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\mathrm{d}x, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\mathrm{d}x. \tag{2.26}$$

These formulas are similar to (2.12) – (2.15) but the limits of integration are different.

EXAMPLE

Consider a piecewise smooth function $f(x)$ defined in the interval $(-L, L)$. Equations (2.22) and (2.23) with $a = -L$ provide the Fourier series for this function and the coefficients of the series as

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{\pi n}{L}x\right) + B_n\sin\left(\frac{\pi n}{L}x\right)\right]. \tag{2.27}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x, \quad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n}{L}x\right)\mathrm{d}x, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n}{L}x\right)\mathrm{d}x. \quad (2.28)$$

These formulas are identical to (2.12) – (2.15) if $L = \pi$.

Lecture 15

EXAMPLE

Find the Fourier series of $x(1-x)$ in the interval $(0, 1)$.

In this example, $a = 0$ and $L = \frac{1}{2}$. Equations (2.24) and (2.25) with $a = 0$ and $L = \frac{1}{2}$ provide the Fourier series for this function and the coefficients of the series as

$$x(1-x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right], \quad (2.29)$$

$$a_0 = \int_0^1 x(1-x)\mathrm{d}x, \quad a_n = 2\int_0^1 x(1-x)\cos(2\pi n x)\mathrm{d}x, \quad b_n = 2\int_0^1 x(1-x)\sin(2\pi n x)\mathrm{d}x.$$

By algebra,

$$a_0 = \frac{1}{6}, \quad a_n = -\frac{1}{\pi^2 n^2}, \quad b_n = 0$$

which make it possible to present (2.29) as

$$x(1-x) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi n x)}{n^2}. \quad (2.30)$$

Setting $x = 0$ in (2.30), we find

$$0 = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

or

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

FOURIER SERIES OF EVEN AND ODD FUNCTIONS

Consider a piecewise smooth function $f(x)$ defined in the interval $(-L, L)$. Such a function is called odd if $f(-x) = -f(x)$, and even if $f(-x) = f(x)$.

If $f(x)$ is odd, then (2.28) give (*check!*)

$$A_n = 0 \;\; (n \geq 0), \quad B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) \mathrm{d}x \;\; (n \geq 1). \tag{2.31}$$

The corresponding Fourier series has the form

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n}{L} x\right). \tag{2.32}$$

If $f(x)$ is even, then (2.28) give (*check!*)

$$B_n = 0 \;\; (n \geq 1), \quad A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi n}{L} x\right) \mathrm{d}x \;\; (n \geq 1), \;\; A_0 = \frac{1}{L} \int_0^L f(x) \mathrm{d}x. \tag{2.33}$$

The corresponding Fourier series has the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n}{L} x\right). \tag{2.34}$$

HALF-RANGE FOURIER SERIES [EK 10.5]

Consider a piecewise smooth function $f(x)$ defined on some interval $(0, L)$. Suppose we extend the domain of definition to $(-L, L)$ by defining the <u>odd function</u>

$$F(x) = f(x) \;\; (0 < x < L) \quad \text{and} \quad F(x) = -f(-x) \;\; (-L < x < 0).$$

According to (2.31) and (2.32), the Fourier series for the odd function $F(x)$ is

$$F(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n}{L} x\right) \quad (-L < x < L), \quad B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) \mathrm{d}x.$$

Limiting the range of x to $(0, L)$, we obtain

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n}{L} x\right) \quad (0 < x < L), \quad B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) \mathrm{d}x. \tag{2.35}$$

EXAMPLE $\qquad f(x) = \cos x \;\; (0 < x < \pi)$

Calculate with $L = \pi$:

$$B_n = \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(nx) \mathrm{d}x = \frac{2}{\pi} [1 - (-1)^{n+1}] \frac{n}{n^2 - 1}.$$

$$B_{2m+1} = 0, \quad B_{2m} = \frac{2}{\pi} \times 2 \times \frac{2m}{4m^2 - 1},$$

$$\boxed{\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m \sin(2mx)}{4m^2 - 1}}.$$

Suppose we extend the domain of definition of the original function $f(x)$ to $(-L, L)$ by defining the <u>even function</u>

$$F(x) = f(x) \ \ (0 < x < L) \quad \text{and} \quad F(x) = f(-x) \ \ (-L < x < 0).$$

According to (2.33) and (2.34), the Fourier series for the even function $F(x)$ is

$$F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n}{L} x\right).$$

Limiting the range of x to $(0, L)$, we obtain

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n}{L} x\right) \quad (0 < x < L), \tag{2.36}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi n}{L} x\right) \mathrm{d}x, \quad A_0 = \frac{1}{L} \int_0^L f(x) \mathrm{d}x.$$

EXAMPLE $\quad f(x) = x^2 \ \ (0 < x < \pi)$

Calculate the coefficients in (2.36) with $L = \pi$:

$$A_0 = \frac{1}{\pi} \int_0^{\pi} x^2 \mathrm{d}x = \frac{\pi^2}{3},$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) \mathrm{d}x = \frac{4}{n^2} (-1)^n.$$

The half-range Fourier series for x^2 is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx). \tag{2.37}$$

Setting $x = 0$, find

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

or in another form

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots = \frac{\pi^2}{12}.$$

Dealing with Fourier series (2.30), we derived the formula

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+...=\frac{\pi^2}{6}.$$

Summing up these two series, we find

$$2+\frac{2}{3^2}+\frac{2}{5^2}+\frac{2}{7^2}+...=\frac{\pi^2}{4}$$

or

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...=\frac{\pi^2}{8}$$

or in another form

$$\boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}}$$

The Fourier series (2.37) can be integrated wrt x term-by-term from $x=0$ to x with the result

$$\frac{x^3}{3}=\frac{\pi^2}{3}x+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nx).$$

Setting in this equality $x=\frac{\pi}{2}$ and taking into account that $\sin(n\pi/2)=(-1)^m$ for $n=2m+1$ and $\sin(n\pi/2)=0$ for $n=2m$, obtain

$$\frac{\pi^3}{3\cdot 8}=\frac{\pi^3}{6}-4\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}$$

or

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}=\frac{\pi^3}{32}$$

or

$$\boxed{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-...=\frac{\pi^3}{32}}$$

It is seen that Fourier series for different functions defined in different intervals provide useful formulas for series in a simple way. However, the Fourier series are very useful for solving boundary value problems for Partial Differential Equations.

Lecture 16

PARTIAL DIFFERENTIAL EQUATIONS [EK 11]

DEFINITION 21 An equation involving one or more partial derivatives of an unknown function of two or more independent variables is called a partial differential equation. The order of the highest derivative is called the order of the equation.

Important linear partial differential equations of the second order

1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t) \quad \underline{\text{hyperbolic equation}}$$

1D heat equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t) \quad \underline{\text{parabolic equation}}$$

2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x,y) \quad \underline{\text{elliptic equation}}$$

2D Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad u = u(x,y) \quad \underline{\text{elliptic equation}}$$

2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad u = u(x,y,t) \quad \underline{\text{hyperbolic equation}}$$

2D heat equation

$$\frac{\partial u}{\partial t} = D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad u = u(x,y,t) \quad \underline{\text{parabolic equation}}$$

DEFINITION 22 A function which has all the partial derivatives appearing in the equation in a region R and satisfies the equation in R is called a solution of the partial differential equation in the region R.

SUPERPOSITION PRINCIPLE If u_1 and u_2 are two solutions of a PDE in some region, then $c_1u_1 + c_2u_2$ is also a solution.

The unique solution of the PDE corresponding to a given physical problem is obtained by using additional information (boundary and initial conditions) specifying the physical situation under consideration.

One-dimensional Wave Equation [Vibrating spring] [EK 11.2]

An elastic string with tension T and mass per unit length ρ is stretched to length L along the x-axis and fixed at the endpoints. The string is distorted and then at $t = 0$ is released and allowed to vibrate in the (x, y) plane. We need to determine the vibrations of the string, this is, to find its deflection $u(x, t)$ at any point $0 < x < L$ and at any time $t > 0$.

Consider a small element of the string $[x, x + \Delta x$ of length Δx. The mass of the element is $\rho\Delta x$. Motion of this element in y-direction is governed by Newton's second law

$$\rho\Delta x \cdot \frac{\partial^2 u}{\partial t^2} = \text{vertical component of the force due to the tension} = F(x,t)$$

Let $\vec{\tau}(x, t)$ be the unit tangent vector to the distorted string, $y = u(x, t)$. Then $F(x, t)$ is the y-component of the force vector

$$T\vec{\tau}(x + \Delta x, t) - T\vec{\tau}(x, t).$$

Recall that $\vec{\tau}(x, t) = (1, u_x)/\sqrt{1 + u_x^2} \approx (1, u_x)$ for small deflections of the string. Therefore,

$$T\vec{\tau}(x + \Delta x, t) - T\vec{\tau}(x, t) \approx T \cdot (1, u_x(x + \Delta x, t)) - T \cdot (1, u_x(x, t)) = T \cdot (0, u_x(x + \Delta x, t) - u_x(x, t)).$$

We have

$$\rho\Delta x \cdot \frac{\partial^2 u}{\partial t^2} = T \cdot (u_x(x + \Delta x, t) - u_x(x, t)).$$

Divide the latter equation by $\rho\Delta x$, denote $c^2 = T/\rho$ and let Δx tend to zero:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{3.1}$$

The constant c in (3.1) is known as the wave speed. This equation is also known as acoustic equation because it describes sound propagation along a tube. The sound speed in air $c = 330$ m/sec.

Show that $u(x, t) = f(\xi)$, $\xi = x - ct$ is a solution of (3.1) for any differentiable function $f(\xi)$.

Show that $u(x, t) = g(\xi)$, $\xi = x + ct$ is also a solution of (3.1) for any differentiable function $g(\xi)$.

Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

describes vibration of a membrane. 3D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

is used in studies of acoustic characteristics of rooms and noise produced by moving (and vibrating) airplanes and ships.

Lecture 17

Heat equation

Consider a solid R. We need to find an equation which governs the temperature distribution $\theta(x, y, z, t)$ inside R. Let us distinguish an arbitrary volume V inside the solid and write down the energy conservation law for this volume. This conservation law states that a variation of thermal energy of the volume V is only due to the flux of the heat energy from outside through the volume boundary ∂V. We assume that there are no sources of the thermal energy inside R.

Variation of the thermal energy of a volume V is calculated as

$$\iiint_V \rho c \Delta\theta(x, y, z, t) \mathrm{d}V,$$

where ρ is the solid density, c is the specific heat capacity, $\Delta\theta$ is the temperature increment in time, $\Delta\theta = \theta(\vec{x}, t + \Delta t) - \theta(\vec{x}, t)$. The specific heat capacity of a material is equal to the energy required to raise the temperature of 1kg of solid by 1C. For water $c = 4.1813 \frac{kJ}{kg \cdot K}$ and for iron $c = 0.45 \frac{kJ}{kg \cdot K}$

If $\vec{q}$ if the flux of the thermal energy (per unit area, per unit time) out of V, then the increase of the thermal energy in V is ($\vec{n}$ is the outer unit normal)

$$-\iint_{\partial V} \vec{q} \cdot \vec{n} \mathrm{d}S \cdot \Delta t.$$

The energy conservation law provides

$$\iiint_V \rho c \Delta\theta(x, y, z, t) \mathrm{d}V = -\iint_{\partial V} \vec{q} \cdot \vec{n} \mathrm{d}S \cdot \Delta t.$$

Dividing this equality by Δt and letting $\Delta t \to 0$, we find

$$\iiint_V \rho c \frac{\partial \theta}{\partial t} \mathrm{d}V = -\iint_{\partial V} \vec{q} \cdot \vec{n} \mathrm{d}S \qquad (a)$$

The divergence theorem gives

$$\iint_{\partial V} \vec{q} \cdot \vec{n} \, \mathrm{d}S = \iiint_V \nabla \cdot \vec{q} \, \mathrm{d}V. \tag{b}$$

Substituting (b) into (a)

$$\iiint_V \left\{ \rho c \frac{\partial \theta}{\partial t} + \nabla \cdot \vec{q} \right\} \mathrm{d}V = 0$$

and taking into account that V is an arbitrary volume in the solid R, we have

$$\rho c \frac{\partial \theta}{\partial t} + \nabla \cdot \vec{q} = 0. \tag{c}$$

The flux of thermal energy $\vec{q}$ is given by Fourier law of heat conduction

$$\vec{q} = -k \nabla \theta, \tag{d}$$

where the constant k is the coefficient of thermal conductivity of the material.

Substituting (d) into (c) and taking into account that $\nabla \cdot \nabla = \nabla^2$, we arrive at the heat equation

$$\frac{\partial \theta}{\partial t} = D \nabla^2 \theta, \tag{3.2}$$

where $D = k/(\rho c)$ is the thermal diffusion coefficient and ∇^2 is the Laplacian

$$\nabla^2 \theta = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}. \tag{3.3}$$

For heat conduction along a 1D bar, when $\theta = \theta(x, t)$, equation (3.3) reduces to

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2}, \tag{3.4}$$

Laplace's equation

Steady state distribution of temperature, when $\theta_t = 0$, is governed by Laplace's equation

$$\nabla^2 \theta = 0, \tag{3.5}$$

as it follows from (3.3). This equation arises in many different contexts: water waves theory, potential flows, electromagnetism.

In 2D case, Laplace's equation reads

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x, y). \tag{3.6}$$

in this unit we concentrate mainly upon solving the 1D wave equation (3.1), 1D heat equation (3.4) and Laplace's equation (3.6).

We will use the Method of Separating Variables to solve each of these equations and Fourier method to satisfy initial and boundary conditions.

Method of Separating Variables for 1D wave equation [EK 11.3]

An elastic string with tension T and mass per unit length ρ is stretched to length L along the x-axis and fixed at the endpoints. The string is distorted and then at $t = 0$ is released and allowed to vibrate in the (x, y) plane. We need to determine the vibrations of the string, this is, to find its deflection $u(x, t)$ at any point $0 < x < L$ and at any time $t > 0$.

The string deflection is governed by the 1D wave equation (3.1)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \quad t > 0). \tag{3.7}$$

We have two boundary conditions

$$u(0, t) = 0, \qquad u(L, t) = 0 \qquad (t > 0). \tag{3.8}$$

The motion of the string depends on the initial deflection $u(x, 0)$ and initial velocity of the deflection $u_t(x, 0)$. We have two initial conditions

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = g(x), \tag{3.9}$$

where the functions $f(x)$ and $g(x)$ are given and such that $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$, this is, the endpoints of the string are fixed.

We need to find the function $u(x, t)$ satisfying equations (3.7) – (3.9).

We shall proceed step by step, as follows.

Step 1. By applying the method of separating variables, we obtain two *ODEs*;

Step 2. We determine solutions of these ODEs, which satisfy the *boundary conditions*;

Step 3. We compose these solutions so that their superposition satisfies the *initial conditions.*

Step 1

We are searching for functions in the form

$$u(x, t) = F(x)G(t) \tag{a}$$

which satisfy the wave equation (3.7).
By inserting (a) into (3.7) we have

$$F(x)G''(t) = c^2 F''(x)G(t). \tag{b}$$

Dividing (b) by c^2FG

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)}. \tag{c}$$

In (c), the LHS depends only on t and the RHS only on x. Therefore, both sides must be equal to a constant k, say. This procedure gives us two ODEs

$$F'' - kF = 0, \tag{d}$$

$$G'' - c^2kG = 0. \tag{e}$$

Step 2

We shall now determine solutions of equations (d) and (e) such that the function (a) satisfies the boundary conditions (3.8). These conditions should be satisfied for any $t > 0$. This is possible if and only if

$$F(0) = 0, \qquad F(L) = 0. \tag{f}$$

This implies that we should find solutions of H-LODE2 (d) subject to homogeneous boundary conditions (f). Note that $F = 0$ is a solution of this homogeneous BVP. The question is *Are there non-trivial solutions and for which values of k these solutions can be obtained?*

Up to now the constant k in (d) was arbitrary. Let's consider three different cases.

First, $k = 0$. In this case

$$F(x) = c_1x + c_0$$

and (f) provides $c_1 = 0$ and $c_2 = 0$. Thus, we have only the trivial solution if $k = 0$.

Second, k is positive. In this case, $k = \mu^2$, $\mu > 0$, and the general solution of (d) is

$$F(x) = c_1 \exp(\mu x) + c_2 \exp(-\mu x).$$

Check that the boundary conditions f can be satisfied only with $c_1 = 0$ and $c_2 = 0$.

Therefore, for $k \geq 0$ in (d) there are no non-trivial solutions which satisfy the boundary conditions (f).

In the third case k is negative, say, $k = -\mu^2$. A general solution of (d) reads

$$F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x) \tag{g}$$

Substituting (g) into the boundary condition $F(0) = 0$, we find $c_1 = 0$. The boundary condition $F(L) = 0$ provides

$$c_2 \sin(\mu L) = 0. \tag{h}$$

If $c_2 = 0$ in (h), then $F(x) = 0$, which is the trivial solution. Hence $\sin(\mu L) = 0$ which gives

$$\mu_n = n\pi/L, \qquad n = 1,\ 2,\ 3,\ .. \tag{i}$$

Setting $c_2 = 1$, we obtain infinitely many solutions

$$F_n(x) = \sin(\pi n x/L). \tag{j}$$

Substituting (i) into (e), we find a general solution of this equation

$$G(t) = A_n \cos(cn\pi t/L) + B_n \sin(cn\pi t/L) \tag{k}$$

with undetermined coefficients A_n and B_n. Equations (a), (j) and (k) provide the functions

$$u_n(x,t) = \left[A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right] \sin\left(\frac{\pi n}{L}x\right) \tag{l}$$

which satisfy the wave equation (3.7) and the boundary conditions (3.8).

Step 3

The solutions (l) cannot satisfy the initial conditions (3.9) on their own. The Superposition Principle shows that the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \tag{m}$$

also satisfy the wave equation (3.7) and the boundary conditions (3.8).

Substitute (m) and (e) into (3.9):

$$\sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi}{L}0\right) + B_n \sin\left(\frac{cn\pi}{L}0\right)\right] \sin\left(\frac{\pi n}{L}x\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n}{L}x\right) = f(x), \tag{n}$$

$$\sum_{n=1}^{\infty} \left[-A_n \frac{cn\pi}{L} \sin\left(\frac{cn\pi}{L}0\right) + B_n \frac{cn\pi}{L} \cos\left(\frac{cn\pi}{L}0\right)\right] \sin\left(\frac{\pi n}{L}x\right) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{L} \sin\left(\frac{\pi n}{L}x\right) = g(x). \tag{o}$$

It is seen that (n) and (o) are half-range sin-Fourier series (2.35). Therefore, formulas (2.35) give

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{\pi n}{L}x\right) \mathrm{d}x, \qquad B_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{\pi n}{L}x\right) \mathrm{d}x. \tag{p}$$

EXAMPLE [EK, 11.3]

Find the shape of the string corresponding to the triangular initial deflection

$$f(x) = 2ax/L \quad (0 < x < L/2), \quad f(x) = 2a(L - x)/L \quad (L/2 < x < L),$$

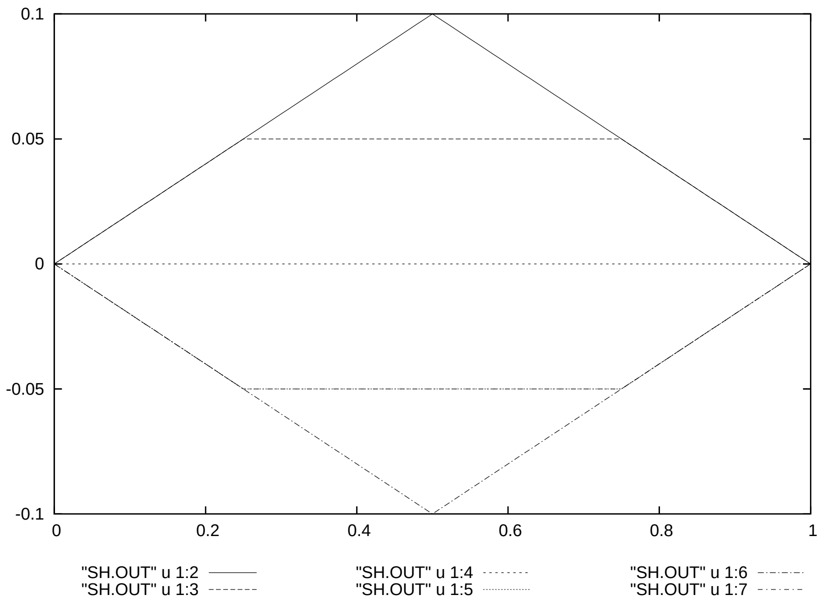
where a is the amplitude of the initial deflection. The string deflection is described by the wave equation with $c^2 = T/\rho$. The string length is L. There is no initial velocity of the string, $g(x) = 0$.

Using formulae (l), (m) and (p), show that

$$u(x,t) = \frac{8a}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cos[\pi c(2m+1)t/L] \sin[\pi x(2m+1)/L] \tag{3.10}$$

The string shapes for $c = 1$m/s, $L = 1$m and $a = 10$cm at t=0s, 0.25s, 0.5s, 0.75s, 1.0s are shown in the following Figures with 100 and 10 terms taken in the series (3.10).

It is seen that the solution is well reproduced even with small number of terms.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0
-0.05
-0.1
0
0.2
0.4
0.6
0.8
1
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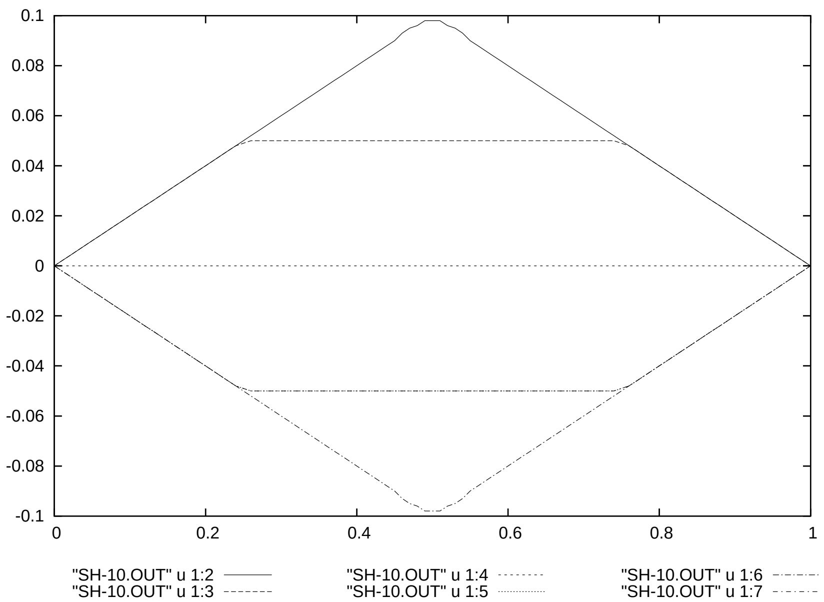
"SH.OUT" u 1:3
"SH.OUT" u 1:4
"SH.OUT" u 1:5
"SH.OUT" u 1:6
"SH.OUT" u 1:7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0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
0.2
0.4
0.6
0.8
1
"SH-10.OUT" u 1:2
"SH-10.OUT" u 1:3
"SH-10.OUT" u 1:4
"SH-10.OUT" u 1:5
"SH-10.OUT" u 1:6
"SH-10.OUT" u 1:7

Lecture 18

Method of Separating Variables for 1D heat equation [EK 11.5]

The temperature distribution in a long thin bar of constant cross section and homogeneous material is described by the one-dimensional heat equation (3.4)

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}, \tag{3.11}$$

The bar of length L is oriented along the x-axis. The ends of the bar are kept at temperature zero. This gives us the boundary conditions

$$\theta(0,t) = 0, \quad \theta(L,t) = 0. \tag{3.12}$$

Let $f(x)$ be the initial distribution of the temperature in the bar. Then the initial condition is

$$\theta(x,0) = f(x). \tag{3.13}$$

Note that the number of initial conditions is equal to the highest order of the time derivatives in the equation. Two initial conditions were required for the wave equation but one for the heat equation.

We shall determine a solution $\theta(x,t)$ satisfying (3.11), (3.12) and (3.13).

FIRST STEP.

We are searching for functions in the form

$$\theta(x,t) = F(x)G(t) \tag{3.14}$$

which satisfy the heat equation (3.11).

By inserting (3.14) into (3.11) we have

$$F(x)G'(t) = DF''(x)G(t). \tag{3.15}$$

Dividing (3.15) by DFG

$$\frac{G'(t)}{DG(t)} = \frac{F''(x)}{F(x)}. \tag{3.16}$$

In (3.16), the LHS depends only on t and the RHS only on x. Therefore, both sides must be equal to a constant k, say. This procedure gives us two ODEs

$$F'' - kF = 0, \tag{3.17}$$

$$G' - DkG = 0, \tag{3.18}$$

SECOND STEP.

We shall now determine solutions of equations (3.17) and (3.18) such that the function (3.14) satisfies the boundary conditions (3.12). These conditions should be satisfied for any $t > 0$. This is possible if and only if

$$F(0) = 0, \qquad F(L) = 0. \tag{3.19}$$

The problem (3.17), (3.19) is identical to the problem (d), (f) for the wave equation. Therefore, non-trivial solutions of problem (3.17), (3.19) exist only for $k = -(\pi n/L)^2$, $n \geq 1$, and have the form

$$F_n(x) = \sin\left(\frac{\pi n}{L}x\right). \tag{3.20}$$

A general solution of ODE (3.18) with $k = -(\pi n/L)^2$ is

$$G_n(t) = A_n \exp\left\{-\left(\frac{\pi n}{L}\right)^2 Dt\right\} \tag{3.21}$$

Equations (3.14), (3.20) and (3.21) provide the functions

$$\theta_n(x,t) = A_n \exp\left\{-\left(\frac{\pi n}{L}\right)^2 Dt\right\} \sin\left(\frac{\pi n}{L}x\right) \tag{3.22}$$

which satisfy the heat equation (3.11) and the boundary conditions (3.12) for any coefficients A_n.

THIRD STEP.

The solutions (3.22) cannot satisfy the initial conditions (3.13) on their own. The Superposition Principle shows that the linear combination

$$\theta(x,t) = \sum_{n=1}^{\infty} \theta_n(x,t) \tag{3.23}$$

also satisfies the heat equation (3.11) and the boundary conditions (3.12).

Substitute (3.23) and (3.22) into the initial condition (3.13):

$$\sum_{n=1}^{\infty} A_n \exp\left\{-\left(\frac{\pi n}{L}\right)^2 D\cdot 0\right\}\sin\left(\frac{\pi n}{L}x\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n}{L}x\right) = f(x), \tag{3.24}$$

This equation is identical to equation (n) for the string problem.

It is seen that (3.24) is half-range sin-Fourier series (2.35). Therefore, formulas (2.35) give

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n}{L}x\right)\,\mathrm{d}x \qquad (3.25).$$

Formula (3.23) with (3.22) and (3.25) provides the required solution of the problem (3.11) – (3.13).

EXAMPLE [EK, 11.5]

Find the temperature evolution in a bar of length L whose ends are kept at zero temperature, $\theta(0,t) = 0$ and $\theta(0,t) = 0$. Initial temperature distribution in the bar $\theta(x,0) = f(x)$ is triangular

$$f(x) = 2ax/L \quad (0 < x < L/2), f(x) = 2a(L-x)/L \quad (L/2 < x < L)$$

where a is a given constant.

Compare the solutions of the wave equation and the heat equation to deduce

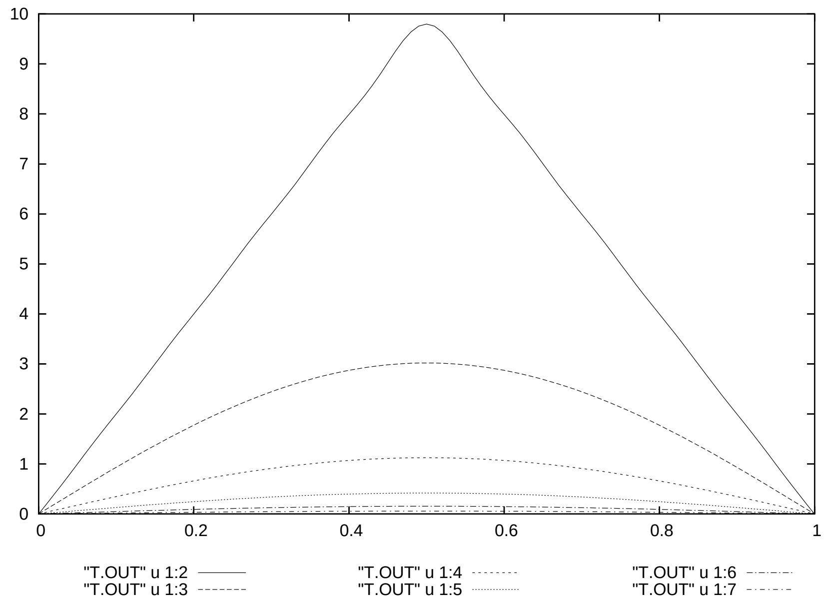
$$\theta(x,t) = \frac{8a}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\exp\left\{-\left(\frac{\pi(2m+1)}{L}\right)^2 Dt\right\}\sin[\pi x(2m+1)/L] \tag{3.26}$$

The temperature distributions for $D = 1\mathrm{m}^2/\mathrm{s}$, $L = 1\mathrm{m}$ and $a = 10\mathrm{C}$ at t=0s, 0.1s, 0.2s, 0.3s are shown in the following Figures with 10 terms taken in the series (3.26).

IMPORTANT

If the initial temperature distribution is given as $f(x) = \theta_0 \sin(\pi x/L)$, then (3.24) provides $A_1 = \theta_0$ and $A_n = 0$ for $n \geq 2$. Correspondingly, $\theta(x,t) = \theta_1(x,t)$ in (3.23), where

$$\theta_1(x,t) = \theta_0 \exp\left\{-\left(\frac{\pi}{L}\right)^2 Dt\right\}\sin\left(\frac{\pi}{L}x\right).$$



Method of Separating Variables for 2D Laplace's equation [EK 11.5]

If the temperature does not change in time, then the steady temperature distribution $\theta(x, y)$ in a 2D solid R is governed by Laplace's equation

$$\nabla^2\theta = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0. \tag{3.27}$$

We should determine a solution of (3.27) which satisfies a given boundary condition on the boundary curve C of 2D solid R. There are three types of the boundary conditions:

1. *Dirichlet BC*, if θ is prescribed on C;

2. *Neumann BC*, if the normal derivative $\partial\theta/\partial n$ is prescribed on C;

2. *mixed boundary conditions*, if θ is prescribed on a portion of C and $\partial\theta/\partial n$ on the rest of C.

We consider a Dirichlet problem in a rectangular R, assuming that the temperature θ is equal to a given function $f(x)$ on the upper side, $y = b$, $0 < x < L$, and zero on the rest of the rectangular boundary:

$$\theta(x, b) = f(x) \quad (0 < x < L), \qquad \theta(x, 0) = 0 \quad (0 < x < L), \tag{3.28}$$

$$\theta(0, y) = 0 \quad (0 < y < b), \qquad \theta(L, y) = 0 \quad (0 < y < b), \tag{3.29}$$

We shall determine a solution $\theta(x, y)$ satisfying (3.27), (3.28) and (3.29).

FIRST STEP.

We are searching for functions in the form

$$\theta(x, y) = F(x)G(y) \tag{3.30}$$

which satisfy Laplace's equation (3.27).

By inserting (3.30) into (3.27) we have

$$F''(x)G(y) + F(x)G''(y) = 0. \tag{3.31}$$

Dividing (3.31) by FG

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}. \tag{3.32}$$

In (3.32), the LHS depends only on x and the RHS only on y. Therefore, both sides must be equal to a constant k, say. This procedure gives us two ODEs

$$F'' - kF = 0, \tag{3.33}$$

$$G'' + kG = 0, \tag{3.34}$$

SECOND STEP.

We shall now determine solutions of equations (3.33) and (3.34) such that the function (3.30) satisfies the boundary conditions (3.28) and (3.29). The conditions (3.29) should be satisfied for any $0 < y < b$. This is possible if and only if

$$F(0) = 0, \qquad F(L) = 0. \tag{3.35}$$

The problem (3.33), (3.35) is identical to the problem (d), (f) for the wave equation. Therefore, non-trivial solutions of problem (3.33), (3.35) exist only for $k = -(\pi n/L)^2$, $n \geq 1$, and have the form

$$F_n(x) = \sin\left(\frac{\pi n}{L} x\right) \tag{3.36}$$

A general solution of ODE (3.34) with $k = -(\pi n/L)^2$ is

$$G_n(y) = A_n \sinh\left\{\frac{\pi n}{L} y\right\} + B_n \cosh\left\{\frac{\pi n}{L} y\right\} \tag{3.37}$$

The boundary condition on $y = 0$ provides $B_n = 0$.

Equations (3.30), (3.36) and (3.37) provide the functions

$$\theta_n(x, y) = A_n \sinh\left\{\frac{\pi n}{L} y\right\} \sin\left(\frac{\pi n}{L} x\right) \tag{3.38}$$

which satisfy Laplace's equation (3.27) and the boundary conditions along the boundary of the rectangular except of its part where $y = b$.

THIRD STEP.

The Superposition Principle shows that the linear combination

$$\theta(x, y) = \sum_{n=1}^{\infty} \theta_n(x, y) \tag{3.39}$$

also satisfies Laplace's equation (3.27) and the boundary conditions along the boundary of the rectangular except of its part where $y = b$.

Substitute (3.39) and (3.38) into the boundary condition at $y = b$:

$$\sum_{n=1}^{\infty} A_n \sinh\left\{\frac{\pi n}{L} b\right\} \sin\left(\frac{\pi n}{L} x\right) = f(x), \tag{3.40}$$

This equation is identical to equation (n) for the string problem.

It is seen that (3.40) is half-range sin-Fourier series (2.35). Therefore, formulas (2.35) give

$$A_n \sinh\left\{\frac{\pi n}{L} b\right\} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) \mathrm{d}x \qquad (3.41).$$

Formulae (3.38) with (3.39) and (3.41) provide the required solution of the *Dirichlet* problem (3.27) – (3.29)

$$\theta(x, y) = \sum_{n=1}^{\infty} a_n \frac{\sinh\left\{\frac{\pi n}{L} y\right\}}{\sinh\left\{\frac{\pi n}{L} b\right\}} \sin\left(\frac{\pi n}{L} x\right), \qquad a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) \mathrm{d}x.$$