Lecture 1

The title of the unit MTH-2C4Y is *"Differential Equations and Algorithms"*. This autumn semester module deals with Differential Equations . Algorithms for numerical solutions of Differential Equations and equations of other types will be considered in the spring semester. A full coverage of the module is offered by the book "Advanced Engineering Mathematics" by Edwin Kreyszig (John Wiley and Sons), Chapters 2, 4, 10 and 11 (about 200 pages in total). Lectures will be twice a week, on Thursdays and Fridays, and will be supported by 3 seminars and four problem classes. Exercises for each seminar will be displayed on Blackboard. Courseworks (2 per seminar) will be included into each Exercise Sheet. Assessment is by examination (80%, 3 questions from 6) and coursework (20%). The courseworks with detailed solutions should be returned by 9th of December, 3pm.

In this autumn module we will mainly deal with a linear differential equation of the second order

$$y'' + p(x)y' + q(x)y = r(x). \tag{0.1}$$

A function $y(x)$ that satisfies (0.1) is termed as solution of (0.1).

The analysis and the methods developed for equation (0.1) will be generalized to differential equations of higher order

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + ...p_1(x)y' + p_0(x)y = r(x). \tag{0.2}$$

We shall learn

(1) how to find the general solutions of the Differential Equations (0.1) and (0.2);
(2) how to solve an initial value problem **IVP** for (0.1), where the values of the unknown function $y(x_0)$ and its first derivative $y'(x_0)$ are specified at some point $x = x_0$;
(3) how to use the method of Frobenius to find a solution to (0.1) in the form of series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha};$$

(4) how to solve a boundary value problem **BVP** for (0.1), where $y(x_0)$ and $y(x_1)$ are known and we need the solution of (0.1) between x_0 and x_1;

We shall study

(5) solutions of Bessel's differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

and Legendre's differential equation

$$(1 - x^2)y'' - 2xy + n(n + 1)y = 0$$

which are important differential equations in applied mathematics. Solutions of these differential equations are known as Bessel functions and Legendre polynomials.
(6) periodic solutions to equation (0.1), $y(x+T) = y(x)$, where T is a period, for a periodic right-hand side (**RHS**) $r(x)$ with the help of Fourier Series.

Finally, we shall find

(7) solutions of the Elementary Partial Differential Equations

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \nabla^2 u \qquad \text{(wave equation)}$$
$$\frac{\partial u}{\partial t} = k \nabla^2 u \qquad \text{(heat equation)}$$
$$\nabla^2 u = 0 \qquad \text{(Laplace's equation)}$$

obtaining them by separation of variables in rectangular Cartesian co-ordinates (Fourier analysis), in cylindrical (Bessel functions are involved) and spherical (Legendre polynomials are involved) co-ordinates. These solutions describe, in particular, the temperature distributions in solids of different shapes.

Topics (1) – (4) will be covered in 6 lectures, topics (5) and (6) in 4 lectures each and topic (7) will be covered in 6 lectures.

First seminar will be in week 4 and will be based on topics (1) – (4). Second seminar in week 7 will cover topic (5) and some elements of the Fourier analysis from (6). Third seminar will be about the method of separation (7) and advanced Fourier analysis from (6).

Modeling via Differential Equations

How to translate a physical phenomenon into a set of equations which describes it?

Step 1: Clearly state the assumptions on which the model will be based. These assumptions should describe the relationships between the quantities to be studied. Usually the assumptions are based upon some observations and experiments.

Step 2: Completely describe the parameters and variables to be used in the model.

Step 3: Derive mathematical equations relating the parameters and variables.

Consider a vertical spring with upper end fixed and a body attached to lower end. If we pull the body down a certain distance from its static position and then release it, the body undergoes a motion. The body is assumed to move strictly vertically.

Step 1: Mass of the spring is disregarded compared to the mass m of the body. According to Newton's second law, the body motion is governed by balance between the inertia force $my''(t)$, where $y(t)$ is the body displacement from the static position, $y = 0$, with the positive direction downwards, and t is time, spring force $F_s(t)$, damping force $F_d(t)$ and a given external force $F_e(t)$. Experiments show that within reasonable limits, the spring force is proportional to the change in the spring length, $F_s = -ky(t)$, which is known as Hooke's law, and the damping force is proportional to the body velocity, $F_d = -cy'(t)$, where k is called the spring stiffness coefficient and c is called the damping coefficient. Here, $k > 0$ and $c > 0$. Initial displacement of the body from its static position is y_0 and initial velocity of the body is equal to zero.

Step 2: We should obtain the body displacement $y(t)$ as a function of time t for given parameters of the problem m, k, c, y_0 and for given external force $F_e(t)$.

Step 3: Newton's second law provides the differential equation with respect to the body displacement

$$my'' = -cy'(t) - ky(t) + F_e(t)$$

or

$$y'' + (c/m)y' + (k/m)y = F_e(t)/m.$$

The latter equation has the form (0.1). If the body mass varies in time, $m = m(t)$, or the spring becomes weaker in time due to aging effect, $k = k(t)$, then the coefficients in the latter equation should be considered as known functions of time.

[*see EK, Section 2.6 for more details and solutions*]

Definitions and Basic Theorems

DEFINITION 1 A differential equation is an equation involving independent variables, an unknown function and its derivatives. Independent variables are real and involved functions are real and smooth.

In (0.1), x is the real independent variable, $y(x)$ is the unknown real function with its first and second derivatives being continuous, involved functions are $y(x), p(x), q(x), r(x)$. The involved functions are assumed real and continuous.

DEFINITION 2 The order of a differential equation is the order of the highest derivative of the unknown function involved in the equation.

Lecture 2

DEFINITION 3 A differential equation is said to be ordinary (**ODE**) if there is just one independent variable. Otherwise, a differential equation is said to be partial differential equation (**PDE**).

Equation (0.1) is ODE of the 2nd order.

Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ is PDE of the 2nd order.

We do not study systems of differential equations in this module.

DEFINITION 4 A linear ODE of order n is a differential equation written in the form (0.2).

Equation (0.1) is linear ordinary differential equation of the second order (**LODE2**).

Equation $y'' = \sqrt{y_x^2 + 1}$ is a nonlinear ODE.

We do not study nonlinear differential equations in this module.

Note that other notations for derivatives can be used. For example,

$$y''' = y^{(3)} = \mathrm{d}^3 y/\mathrm{d}x^3 = y_{xxx}.$$

In all our considerations, we assume that the independent variable x varies in some given finite interval $I = (a, b)$ or on the entire x- axis, $I = (-\infty, +\infty)$.

DEFINITION 5 Functions $p(x)$, $q(x)$ in the LODE2 (0.1) and the functions $p_j(x)$, $j = 0, 1, 2, ..., n$ in the LODEn (0.2) are called the coefficients of these equations.

DEFINITION 6 The LODEn (0.2) is said to be written in standard form, if $p_n(x) \equiv 1$.

Equation (0.1) is written in standard form.

DEFINITION 7 A function $y = \phi(x)$ is called a solution of the LODEn (0.2) on interval I (perhaps infinite), if $\phi(x)$ is defined and n times differentiable throughout I and is such that equation (0.2) becomes an identity for any $x \in I$ when we replace $y(x)$ and its derivatives in (0.2) by $\phi(x)$ and its corresponding derivatives.

EXAMPLE The function $y(x) = \sin(\omega x)$ is a solution of the LODE2

$$y'' + \omega^2 y = 0, \tag{0.3}$$

where ω is constant. Indeed,

$y'(x) = \omega\cos(\omega x)$, $y''(x) = -\omega^2\sin(\omega x)$ and

$$-\omega^2\sin(\omega x) + \omega^2\sin(\omega x) \equiv 0!$$

The functions $\cos(\omega x)$, $c_1\sin(\omega x)$, $c_2\cos(\omega x)$ and $c_1\sin(\omega x)+c_2\cos(\omega x)$ are also solutions of equation (0.3). (*Check by substitution*)

IMPORTANT! **Solution of a differential equation is not unique.**

The function $y(x) = \sin(\omega x)$ is not a solution of the LODE2

$$y'' + \omega^2 y = 1.$$

DEFINITION 8 LODE (0.2) is said to be homogeneous (H) if its right-hand side (**RHS**) $r(x)$ is zero for all $x \in I$. If $r(x) \neq 0$, then (0.2) is said no be nonhomogeneous (NH).

To the NH-LODE2 (0.1), we associate the so called associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{0.4}$$

THEOREM 1

(i) If $y = y_1(x)$ is a solution of the HLODE (0.4) on some interval I, then $y = c_1y_1(x)$ is also a solution of (0.4).

(ii) If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of (0.4) on I, then their linear superposition

$$y = c_1y_1(x) + c_2y_2(x) \tag{0.5}$$

is also a solution for any arbitrary constants c_1 and c_2.

PROOF Substitute (0.5) in equation (0.4), collect the terms with c_1 and c_2, and recall that $y = y_1(x)$ and $y = y_2(x)$ are solutions of this equation. ¶

IMPORTANT! **THEOREM 1 does not hold for nonhomogeneous LODE.**

DEFINITION 9 A general solution of HLODE (0.4) on I is a function of the form (0.5), where $y = y_1(x)$ and $y = y_2(x)$ are linearly independent (not proportional) solutions of (0.4).

Any particular solution of the HLODE (0.4) on I is obtained by assigning specific values to the constants c_1 and c_2 in (0.5).

A general solution includes all possible solutions of (0.4).

Two linearly independent (not proportional) solutions $y = y_1(x)$ and $y = y_2(x)$ are said to form a basis (or fundamental system) of solutions of equation (0.4).

DEFINITION 10 Let $y = y_1(x)$ and $y = y_2(x)$ be two differentiable functions defined on an interval I. We say that these functions are proportional (linearly dependent) if and only if there exists a constant C such that

$$y_2(x) = Cy_1(x) \quad \text{for any} \quad x \in I.$$

How to distinguish two linearly independent solutions of equation (0.4) in order to find the general solution (0.5)?

The following statements are equivalent:

1) $y = y_1(x)$ and $y = y_2(x)$ are proportional;

2) $y_2(x)/y_1(x)$ is constant;

3) $[y_2(x)/y_1(x)]' = 0$;

4) $y_1y_2' - y_1'y_2 = 0$.

Therefore, $y = y_1(x)$ and $y = y_2(x)$ are not proportional on I if and only if $y_1y_2' - y_1'y_2 \neq 0$ for all $x \in I$.

DEFINITION 11 The Wronskian $W(y_1, y_2)(x)$ is defined as

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

THEOREM 2

Two functions $y = y_1(x)$ and $y = y_2(x)$ are linearly independent on I if and only if $W(y_1, y_2)(x) \neq 0$ for all $x \in I$.

Lecture 3

EXAMPLE: $y_1(x) = \sin(\omega x)$ and $y_2(x) = \cos(\omega x)$ are linearly independent:

$$W(y_1, y_2)(x) = \begin{vmatrix} \sin(\omega x) & \cos(\omega x) \\ \omega\cos(\omega x) & -\omega\sin(\omega x) \end{vmatrix} = -\omega\sin^2(\omega x) - \omega\cos^2(\omega x) =$$
$$-\omega\left[\sin^2(\omega x) + \cos^2(\omega x)\right] = -\omega \neq 0!$$

THEOREM 3 (Liouville-Abel formula)

If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of (0.4) on I, then

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\exp\left(-\int_{x_0}^{x} p(t)\mathrm{d}t\right), \tag{0.6}$$

where $x_0 \in I$.

PROOF Calculate

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\Big(y_1y_2' - y_1'y_2\Big) = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' = y_1y_2'' - y_2y_1'' =$$
$$y_1\Big[-py_2' - qy_2\Big] - y_2\Big[-py_1' - qy_1\Big] = -p(x)\Big(y_1y_2' - y_1'y_2\Big) = -p(x)W(x).$$

Therefore,

$$\frac{\mathrm{d}W}{\mathrm{d}x} = -p(x)W(x).$$

This first-order differential equation is solved by separation of variables

$$\int \frac{\mathrm{d}W}{W} = -\int p(x)\mathrm{d}x$$

which provides (0.6).¶

RESULT Theorems 2 and 3 give

Two solutions $y = y_1(x)$ and $y = y_2(x)$ are linearly independent (not proportional) on $I \Leftrightarrow$ (Th2) $W(y_1, y_2)(x) \neq 0$ for every $x \in I \Leftrightarrow$ (Th3) $W(y_1, y_2)(x_0) \neq 0$ at a single point of I.

IMPORTANT Functions $y = y_1(x)$ and $y = y_2(x)$ in Theorem 3 are not arbitrary. They are solutions of the HLODE (0.4).

The general solution of (0.4) has the form (0.5), where $y = y_1(x)$ and $y = y_2(x)$ are linearly independent. In order to specify the coefficients c_1 and c_2 in this solution, we need two additional conditions. In many applications the following conditions are used

$$y(x_0) = K_0, \quad y'(x_0) = K_1, \tag{0.7}$$

where K_0 and K_1 are given numbers. The conditions (0.7) are called initial conditions. Homogeneous LODE equation (0.4) or the nonhomogeneous LODE (0.1) together with the initial conditions (0.7) are known as an initial value problem (**IVP**).

THEOREM 4 Suppose that the coefficients $p(x)$ and $q(x)$ in the HLODE (0.4) are continuous on I and $x_0 \in I$. Then the IVP (0.4), (0.7) has a unique solution on I for any K_0 and K_1.

The uniqueness of the solution of the IVP (0.4), (0.7) is proved in EK, Section "Further proof" in Chapter 2, pp 148-149.¶

THEOREM 5 If the coefficients $p(x)$ and $q(x)$ in (0.4) are continuous on some interval I, then the HLODE (0.4) has a general solution on I.

PROOF Let $x_0 \in I$. Consider two functions $y = y_1(x)$ and $y = y_2(x)$ which are the solutions of the IVP for equation (0.4) with the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0,$$

$$y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

According to Theorem 4, these functions are uniquely defined.

Calculate the Wronskian $W(y_1, y_2)(x_0)$ at $x = x_0$:

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, $y = y_1(x)$ and $y = y_2(x)$ are linearly independent on I.

Consider a particular solution $y = Y(x)$ of equation (0.4) on I. Let us prove that this solution can be presented as a linear superposition of the functions $y = y_1(x)$ and $y = y_2(x)$.

Calculate $Y(x_0)$ and $Y'(x_0)$ and formulate the IVP

$$y'' + p(x)y' + q(x)y = 0 \quad (x \in I),$$

$$y(x_0) = Y(x_0), \quad y'(x_0) = Y'(x_0).$$

The solution of this IVP is unique (see Theorem 4) and, therefore, has to be equal to $Y(x)$ on I. On the other hand, the solution of this IVP can be presented as (check!)

$$y(x) = Y(x_0)y_1(x) + Y'(x_0)y_2(x).$$

Therefore, any solution $y = Y(x)$ of (0.4) can be presented as (0.5), where $c_1 = Y(x_0)$ and $c_2 = Y'(x_0)$.¶

RESULT In order to build a general solution of HLODE (0.4), we need to find two linearly independent solutions $y = y_1(x)$ and $y = y_2(x)$ of this equation.

Lecture 4

NONHOMOGENEOUS DIFFERENTIAL EQUATION

RESULT The general solution $y_{GNH}(x)$ of the nonhomogeneous LODE (0.1) is given by

$$y_{GNH}(x) = y_{GH}(x) + y_{PNH}(x), \tag{0.8}$$

where

$y_{GH}(x)$ is a general solution of the associated homogeneous equation (which is the equation (0.4)!!!);

$y_{PNH}(x)$ is a particular solution of the nonhomogeneous equation (0.1).

How to find a particular solution if the general solution of the associated homogeneous equation is known?

METHOD OF VARIATION OF PARAMETERS (EK 2.16)

A general solution of the associated homogeneous equation (0.4) on I has the form

$$y_{GH}(x) = c_1 y_1(x) + c_2 y_2(x), \tag{0.9}$$

where $y = y_1(x)$ and $y = y_2(x)$ are assumed known functions. The method of variation of parameters consists in replacing c_1 and c_2 by functions $u(x)$ and $v(x)$ to be determined so that the resulting function

$$y_{PNH}(x) = u(x)y_1(x) + v(x)y_2(x) \tag{0.10}$$

is a particular solution of the nonhomogeneous equation (0.1).

By differentiating (0.10) we obtain

$$y'_{PNH}(x) = u'(x)y_1(x) + u(x)y'_1(x) + v'(x)y_2(x) + v(x)y'_2(x).$$

Let us impose that $u(x)$ and $v(x)$ satisfy the equation

$$u'(x)y_1(x) + v'(x)y_2(x) = 0. \tag{0.11}$$

Then the first derivative reads

$$y'_{PNH}(x) = u(x)y'_1(x) + v(x)y'_2(x).$$

By differentiating this function we find

$$y''_{PNH}(x) = u'(x)y'_1(x) + u(x)y''_1(x) + v'(x)y'_2(x) + v(x)y''_2(x).$$

By substituting the calculated derivatives into (0.1) and collecting terms with $u(x)$ and $v(x)$, we obtain

$$u(x)[y''_1 + p(x)y'_1 + y_1] + v(x)[y''_2 + p(x)y'_2 + y_2] + u'(x)y'_1 + v'(x)y'_2 = r(x).$$

Since $y = y_1(x)$ and $y = y_2(x)$ are solutions of the homogeneous equation (0.4), the latter equation and (0.11) provide the system of algebraic equations with respect to $u'(x)$ and $v'(x)$:

$$u'(x)y_1(x) + v'(x)y_2(x) = 0,$$

$$u'(x)y'_1(x) + v'(x)y'_2(x) = r(x).$$

The solution of this system is

$$u'(x) = -y_2(x)r(x)/W(x), \quad v'(x) = y_1(x)r(x)/W(x),$$

where $W(x) = W(y_1, y_2)(x)$ is the Wronskian of $y_1(x)$ and $y_2(x)$. It is important to note that $W(x) \neq 0$ since $y_1(x)$ and $y_2(x)$ constitute a basis of solutions.

Integrating these equations, we obtain

$$u(x) = -\int \frac{y_2(x)r(x)}{W(x)}\mathrm{d}x, \qquad v(x) = \int \frac{y_1(x)r(x)}{W(x)}\mathrm{d}x.$$

Substituting these expressions for $u(x)$ and $v(x)$ into (0.10), we obtain a solution of the nonhomogeneous equation

$$y_{PNH}(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(x)}\mathrm{d}x + y_2(x)\int \frac{y_1(x)r(x)}{W(x)}\mathrm{d}x. \tag{0.12}$$

IMPORTANT Before applying (0.12) make sure that the Differential Equation is written in the standard form.

IMPORTANT Do not mix $y_1(x)$ and $y_2(x)$ in (0.12) and calculations of Wronskian $W(y_1, y_2)(x)$.

IMPORTANT RESULT If two linearly independent solutions of the H-LODE (0.4) $y_1(x)$ and $y_2(x)$ are known, then a general solution of the NH-LODE (0.1) is given by formulae (0.8), (0.9) and (0.12).

REDUCTION OF ORDER

If a solution $y_1(x)$ of the H-LODE (0.4) is known, then a second linearly independent solution $y_2(x)$ can be found by solving a first-order differential equation .

A second solution of (0.4) is sought in the form

$$y = u(x)y_1(x), \tag{0.13}$$

where $u(x)$ is a new unknown function. By substituting (0.13) into H-LODE (0.4) and collecting terms with u, u' and u'', we find

$$u''y_1 + u'(2y_1' + py_1) + u[y_1'' + py_1' + qy_1] = 0.$$

The last term on the left-hand side is zero and we arrive at a first-order differential equation with respect to $U(x) = u'(x)$:

$$U' + (2y_1'/y_1 + p(x))U = 0.$$

This first-order differential equation is solved by separation of variables as

$$\frac{\mathrm{d}U}{U} = -\frac{2\mathrm{d}y_1}{y_1} - p(x)\mathrm{d}x$$

with the solution

$$U(x) = C_1 y_1^{-2}(x)\exp\Big(-\int p(x)\mathrm{d}x\Big)$$

where C_1 is an arbitrary constant which can be taken to be unity. Finally we integrate the differential equation $u'(x) = U(x)$ and substitute the result into (0.13)

$$y_2(x) = y_1(x)\int y_1^{-2}(x)\exp\Big(-\int p(x)\mathrm{d}x\Big)\mathrm{d}x. \tag{0.14}$$

IMPORTANT $y_2(x)$ is not proportional to $y_1(x)$ because $y_2(x)/y_1(x) = u(x) \neq Const$ (*Why?*)

RESULT If a solution $y_1(x)$ of the associated homogeneous equation (0.4) is known, then a general solution of the NH-LODE (0.1) can be written using formulae (0.12) and (0.14).

Lecture 5

EXAMPLE: Find a general solution of the nonhomogeneous Legendre equation ($n = 1$)

$$(1-x^2)y'' - 2xy' + 2y = 1 \quad (-1 < x < 1), \tag{0.15}$$

for which $y_1 = x$ is a solution of the associated homogeneous differential equation .

IMPORTANT Equation (0.15) is not written in standard form.

A general solution of NH-LODE is given by (0.8).

Check that $y_1 = x$ is a solution of the homogeneous equation $(1-x^2)y'' - 2xy' + 2y = 0$.

Check that $y_{PNH}(x) = 1/2$ a particular solution of (0.15).

In order to find a second solution of the associated homogeneous equation, write this equation in the standard form (divide by $1 - x^2$):

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0$$

and use (0.14) with $y_1 = x$ and $p(x) = -\frac{2x}{1-x^2}$:

$$y_2(x) = x\int x^{-2}\exp\Big(\int\frac{2x\mathrm{d}x}{1-x^2}\Big)\mathrm{d}x = x\int\frac{\mathrm{d}x}{x^2(1-x^2)} = -1 + \frac{x}{2}\log\Big(\frac{1+x}{1-x}\Big).$$

A general solution of NH-LODE (0.15) is

$$y_{GNH}(x) = c_1 x + c_2\Big[-1 + \frac{x}{2}\log\Big(\frac{1+x}{1-x}\Big)\Big] + \frac{1}{2}.$$

EXAMPLE: Find the solution of the IVP for equation (0.15) with the initial conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

Calculate by using the general solution $y_{GNH}(x)$:

$$y_{GNH}(0) = -c_2 + \frac{1}{2} \quad \Longrightarrow \quad c_2 = -\frac{1}{2}.$$

$$y'_{GNH}(x) = c_1 + \frac{c_2}{2}\Big[\log\Big(\frac{1+x}{1-x}\Big) + x\Big(\frac{1}{1+x} + \frac{1}{1-x}\Big)\Big],$$

$$y'_{GNH}(0) = c_1 \quad \Longrightarrow \quad c_1 = 0.$$

The solution of the IVP reads

$$y(x) = 1 - \frac{x}{4}\log\Big(\frac{1+x}{1-x}\Big).$$

EXAMPLE: Find the solution of the Boundary Value Problem (**BVP**)for equation (0.15) with the boundary conditions

$$y(0) = 1, \qquad y(\frac{1}{2}) = 0.$$

Calculate by using the general solution $y_{GNH}(x)$:

$$y_{GNH}(0) = -c_2 + \frac{1}{2} \quad \Longrightarrow \quad c_2 = -\frac{1}{2}.$$

$$y_{GNH}(\frac{1}{2}) = \frac{1}{2}c_1 + 1 - \frac{1}{8}\log 3.$$

Substitute $y_{GNH}(\frac{1}{2})$ into the boundary condition $y(\frac{1}{2}) = 0$:

$$\frac{1}{2}c_1 + 1 - \frac{1}{8}\log 3 = 0 \quad \Longrightarrow \quad c_1 = \frac{1}{4}\log 3 - 2$$

The solution of the BVP reads

$$y(x) = 1 + \Big(\frac{1}{4}\log 3 - 2\Big)x - \frac{x}{4}\log\Big(\frac{1+x}{1-x}\Big).$$

Procedure to solve IVP

In order to solve a IVP for NH-LODE2

$$y'' + p(x)y' + q(x)y = r(x) \quad (x \in I), \tag{0.16}$$

$$y(x_0) = K_0, \quad y'(x_0) = K_1 \quad (x_0 \in I), \tag{0.17}$$

where $p(x)$, $q(x)$, $r(x)$ are known continuous functions on I and x_0, K_0, K_1 are given constants, we should

(1) find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous equation;
(2) find a particular solution $y_{PNH}(x)$ of the nonhomogeneous equation (0.16);
(3) present a general solution of (0.16) in the form

$$y_{GNH}(x) = c_1 y_1(x) + c_2 y_2(x) + y_{PNH}(x); \tag{0.18}$$

(4) substitute (0.18) into the initial conditions (0.17)

$$c_1 y_1(x_0) + c_2 y_2(x_0) = K_0 - y_{PNH}(x_0)$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = K_1 - y_{PNH}'(x_0); \tag{0.19}$$

(5) solve the algebraic system (0.19) with respect to the coefficients c_1 and c_2 and substitute them into (0.18).

This procedure gives us the solution of the IVP (0.16) -(0.17).

Procedure to solve BVP

Consider NH-LODE2 (0.16) on an interval $I = (a, b)$. A general solution of this equation has the form (0.18).

In a BVP, two constants c_1 and c_2 in (0.18) are calculated by using two boundary conditions at the end points of the interval I, $x = a$ and $x = b$:

$$y(a) = K_0, \qquad y(b) = K_1$$

or

$$y'(a) = K_0, \qquad y'(b) = K_1$$

or

$$y(a) = K_0, \qquad y'(b) = K_1$$

and other combinations. General form of boundary conditions is

$$\alpha y(a) + \beta y'(a) = K_0, \qquad \lambda y(a) + \mu y'(a) = K_1, \tag{0.20}$$

where α, β, λ, μ, K_0, K_1 are known constants.

By substituting the general solution (0.18) into the boundary conditions (0.20), we arrive at a system of two linear equations with respect to the coefficients c_1 and c_2 in (0.18). In contrast to a IVP, the solution of a BVP can be not unique even for smooth involved functions or can be not available.

EXAMPLE: Consider LODE2

$$y'' + y = 0$$

on interval $I = (0, \pi)$. Its general solution has the form

$$y_G(x) = c_1 \sin x + c_2 \cos x.$$

Consider the following boundary conditions:

(A)

$$y(0) = 1, \quad y(\pi) = 2 \quad \Longrightarrow c_2 = 1, \quad -c_2 = 2,$$

therefore, there is not solution of this BVP.

(B)

$$y(0) = 1, \quad y(\pi) = -1 \quad \Longrightarrow c_2 = 1, \quad -c_2 = -1,$$

therefore, solution of this BVP is not unique.

(C)

$$y(0) = 0, \quad y'(\pi) = 2 \quad \Longrightarrow c_2 = 0, \quad c_1 = -2,$$

therefore, solution of this BVP is unique, $y(x) = -2\sin x$.

Boundary Value Problems and so-called Eigen Value Problems for LODEs will be considered again in final part of this module, which is on solution of PDEs by the method of separation of variables.