SCHOOL OF MATHEMATICS

MTH-2C4Y/2C41 AUTUMN SEMESTER 2009

DIFFERENTIAL EQUATIONS – EXERCISES 1

Coursework: The exercises marked with asterisk $*$ should be solved with the solutions clearly presented on paper, which must be put into the marked box in room S1.10 before 3pm on 9th of December.

1. Write down a linear ordinary differential equation of order 2 in standard form and explain **all** of the symbols and notations you use.

2. You are given $y_1(x) = \exp(x)$, $y_2(x) = \exp(2x)$. Calculate the Wronskian $W(y_1, y_2)(x)$ and show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent. Explain why the Liouville-Abel formula cannot be used to calculate the Wronskian $W(y_1, y_2)(x)$ in this exercise.

3. Let $y_1(x)$ and $y_2(x)$ be solutions of the differential equation

$$(x+1)y'' + y' + [\log^2|\sin x|]y = 0 \quad (x > -\frac{1}{2}), \qquad (A)$$

which satisfy the initial conditions

$$y_1(0) = 1, \quad y_1'(0) = 0 \qquad\qquad y_2(0) = 0, \quad y_2'(0) = 1.$$

Calculate the Wronskian $W(y_1, y_2)(x)$ by using Liouville-Abel formula in the form

$$W(y_1, y_2)(x) = W(y_1, y_2)(0)\exp\left(-\int_0^x p(t)\mathrm{d}t\right).$$

Deduce a general solution of equation (A) without giving explicit forms of $y_1(x)$ and $y_2(x)$.

4. Find a constant α such that $y_1(x) = x^\alpha$ is a solution of Euler equation

$$x^2y'' - 3xy' + 4y = 0. \qquad (B)$$

Find a second solution of equation (B), $y_2(x)$, using the method of Reduction of Order.
Show that a general solution of equation (B) has the form $y(x) = x^2(c_1 + c_2\log x)$.
Find the solution of the IVP for equation (B) with the initial conditions
$y(1) = 0$, $y'(1) = 2$.

Continued....

5*. (**Total marks: 10**)

Show that $y(x) = x(c_1 + c_2 \log x)$ is a general solution of the homogeneous equation associated with the differential equation

$$x^2 y'' - xy' + y = x^2. \qquad (C)$$

[**2 marks**]

Find a particular solution of the nonhomogeneous equation (C) by the method of variation of parameters. [**5 marks**]

Find the solution of the BVP for equation (C) with the boundary conditions $y(1) = 1$, $y(2) = 4$. [**3 marks**]

6*. (**Total marks: 10**)

Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$ are polynomials of order n, which satisfy the differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0 \quad (-1 \le x \le 1) \qquad (D)$$

By using the Power Series Method show that equation (D) has a polynomial solution of order n. Give formulas for $T_0(x)$, $T_1(x)$ and $T_2(x)$. [**6 marks**]

Find a second solution $y_2(x)$ of equation (D) for $n = 1$, using the method of Reduction of Order. Show that $y_2(x) = \sqrt{1 - x^2}$. [**4 marks**]

7. Use the recurrence relations

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0, \qquad (i)$$

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x), \qquad (ii)$$

where $P_n(x)$ is the Legendre polynomial of degree n, to prove the formula

$$\int P_n(x)\mathrm{d}x = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n + 1} + C.$$

[Hint: Integrate (ii) and account for (i) in the result]