Test 2 Solutions – Summer Evening 2003

Question 1

The set $S = \{a, b, c, d\}$ is a sample space for a particular experiment. The following events are defined on this sample space:

 $E_1 = \{a\}, \quad E_2 = \{a, b\}, \quad E_3 = \{a, b, d\}, \quad E_4 = \{b, c\}.$ If $\Pr[E_1] = .1, \Pr[E_2] = .4$, and $\Pr[E_3] = .8$, what is $\Pr[E_4]$?

We know that the probability of any event is equal to the sum of the probabilities of the sample points included in that event. We are asked to find $Pr[E_4] = Pr[b] + Pr[c]$. Since E_1 contains only the sample point a, then $Pr[E_1] = Pr[a]$, and we see that Pr[a] = .1. Likewise, $Pr[E_2] = Pr[a] + Pr[b]$, so knowing $Pr[E_2]$ allows us to calculate Pr[b] as

$$Pr[b] = Pr[E_2] - Pr[a] = .4 - .1 = .3$$

There are several approaches we can take to find Pr[c], but the easiest is to notice that this is the only sample point *not* included in event E_3 , so we have

$$Pr[c] = Pr[E_3^C] = 1 - Pr[E_3] = 1 - .8 = .2$$

Now we can see that $Pr[E_4] = Pr[b] + Pr[c] = .3 + .2 = .5$. (Answer **C**)

Question 2

A student is selected at random from a particular class and his or her mark (out of 50) on the final exam is looked up in the professor's marks file. (Only integer grades were awarded on the exam.)

- Let S_1 be the set of all students in the class,
- let S_2 be the set of all student numbers of students enrolled in the class, and let S_3 be a subset of the integers from 0 to 50, containing only those which were marks received by students on the final exam.

Which of these sets could be used as an *equiprobable* sample space for this experiment?

Since a student will be selected at random, all students are equally likely to be chosen, so the set of all students in the class is an equiprobable sample space for this experiment. Also, each student number uniquely identifies a particular student, so choosing a student and choosing a student number are effectively the same thing. Therefore we could use the set containing the student numbers of the students in the class, instead of the students themselves, as an equiprobable sample space for the experiment. The set containing the marks which were received by students in the class is certainly a sample space for the experiment, but it is not an *equiprobable* sample space. The various possible marks received may have been achieved by different numbers of students and therefore are not equally likely to be observed when a student is chosen at random. For instance, suppose there were 10 students who all earned a grade of 45 of the exam, and 6 students who earned a grade of 44 on the exam. Then the probability that the selected student earned a grade of 45 is $\frac{10}{n}$, and for a grade of 44 the probability

is $\frac{6}{n}$, where *n* is the number of students in the class, i.e. $n = n(S_1)$, so the probabilities associated with these 2 grades would *not* be equal.

Therefore S_1 and S_2 are both equiprobable sample spaces, but S_3 is not. (Answer **D**)

Question 3

A single poker hand of 5 cards is dealt from a standard deck. What is the probability that this hand contains 3 of a kind (i.e. exactly 3 cards of the same denomination, e.g. exactly 3 tens or exactly 3 aces)?

An equiprobable sample space for this experiment is the set S containing all the different 5-card hands which could be dealt, so we have $n(S) = \binom{52}{5}$. Let E be the event that the hand contains 3 of a kind. There are 13 different denominations which these 3 could all be. That is, we first think about the 'decision' '3 of *what* kind?'. This decision can be made in $\binom{13}{1}$ different ways. Once we know which denomination, the next question is '*which* 3 of them?'. Since there are 4 cards of the chosen denomination, there are $\binom{4}{3}$ ways to pick 3 of them. Finally, we need to consider 'what are the *other* 2 cards in the hand?'. We eliminate the 4 cards of the denomination for which we have already chosen 3, which leaves the other 48 cards in the deck, so there are $\binom{48}{2}$ ways to choose 2 other cards. Therefore we see that $n(E) = \binom{13}{1}\binom{4}{3}\binom{48}{2}$.

Notice: The problem with answer E, in which a card is chosen, then 2 more cards of the same denomination are chosen, and then the other 2 cards are chosen, is that it overcounts by partially ordering the 3 of a kind. It distinguishes between the 'first' one of the 3 and the others, so this answer is 3 times bigger than the correct answer.

(Answer \mathbf{C})

Question 4

A subcommittee of 4 people is selected at random from a committee of 10 people. If Julie and Frances are 2 people on the committee, what is the probability that Julie is chosen for the subcommittee and Frances isn't?

There is $\binom{1}{1} = 1$ way to choose Julie, and $\binom{1}{0} = 1$ way to not choose Frances, and then $\binom{8}{3}$ ways to choose 3 of the other 8 people to be on the committee with Julie. So if E is the event that Julie is chosen for the subcommittee and Frances is not, we have $n(E) = \binom{8}{3}$. Of course, the equiprobable sample space, S, for the experiment is the set of all ways of choosing 4 of the 10 committee members to be on the subcommittee, so $n(S) = \binom{10}{4}$. Therefore we get

$$Pr[E] = \frac{\binom{8}{3}}{\binom{10}{4}} = \frac{\frac{8!}{3!5!}}{\frac{10!}{4!6!}} = \frac{8!}{3!5!} \times \frac{4!6!}{10!}$$
$$= \frac{8!}{3!5!} \times \frac{4 \times 3! \times 6 \times 5!}{10 \times 9 \times 8!} = \frac{4 \times 6}{10 \times 9} = \frac{4}{5 \times 3} = \frac{4}{15}$$

 $(Answer \mathbf{D})$

Laura, Mitch, Norm and Penny arrange themselves randomly around a card table. What is the probability that Mitch and Penny are sitting beside one another?

If S is the set of all ways that the 4 people can sit around the card table, then we have n(S) = (4-1)! = 3! = 6. Let E be the event that Mitch and Penny are sitting beside one another. There are several ways that we can find n(E). The easiest is to have Mitch sit at the table first. It doesn't matter where he sits. Now, we have Penny take either of the 2 seats which are beside him. Finally we have Norm and Laura arrange themselves in the other 2 seats, which can be done in 2! different ways. Therefore $n(E) = 2 \times 2! = 4$, so we have $Pr[E] = \frac{4}{6} = \frac{2}{3}$.

A very different approach to this questions which gets the answer even more quickly is as follows. First, suppose Mitch sits at the table. Now, let S be the set of all places that Penny could sit. Then n(S) = 3. Let E be the event that Penny chooses a seat beside Mitch. Then n(E) = 2 and we see that $Pr[E] = \frac{2}{3}$. (Answer**E**)

Question 6

Let A and B be 2 events defined on a sample space S. If $\Pr[A] = .3$, $\Pr[B] = .6$ and $\Pr[A \cap B] = .2$, what is $\Pr[A|B]$?

We simply use the formula for conditional probability:

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{.2}{.6} = \frac{1}{.3}$$

 $(Answer \mathbf{D})$

Question 7

Let A and B be 2 events defined on a sample space S. If Pr[A] = .3 and Pr[B] = .6, and it is known that A and B are mutually exclusive events, what is Pr[A|B]?

If A and B are mutually exclusive events then $Pr[A \cap B] = 0$ so we have

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{0}{.6} = 0$$

That is, A and B being mutually exclusive events means that they cannot happen together, so if we know that B has occurred, that tells us (for certain) that A has *not* occurred, so the probability that A also occurs is 0. (Answer **A**)

Consider the following game: First, a coin is tossed. If the coin comes up heads, the player wins \$1 and the game is over. If the coin comes up tails, a fair die is rolled. If a 1 or a 2 is rolled, the player wins \$1 and the game is over. If a 3 or a 4 is rolled, the player flips another coin and wins \$3 times the number of heads tossed. If the die roll was a 5 or a 6, then the player draws a single card from a standard deck. If the card is a diamond, the player wins \$10. Otherwise, the player wins \$1. What is the probability that someone who plays this game once wins \$10?

There is only one way for a player to win \$10 in this game. He or she must toss tails, roll a 5 or a 6 and draw a diamond. This happens with probability

$$\frac{1}{2} \times \frac{2}{6} \times \frac{1}{4} = \frac{1}{2 \times 3 \times 4} = \frac{1}{24}$$

If you draw a probability tree modelling this game, the event 'win \$10' occurs on only one path, whose probability is as shown above. (Answer \mathbf{A})

Question 9

Consider the stochastic process modelled by the probability tree shown (on the question paper). What is $\Pr[E \cup F]$?

 $Pr[E \cup F]$ is found by adding up the path probabilities for all paths in which the event $E \cup F$ occurs, i.e. all paths including an E branch, an F branch, or both. There are 3 such paths. We get:

$$Pr[E \cup F] = \left(\frac{1}{4} \times \frac{1}{3}\right) + \left(\frac{1}{4} \times \frac{1}{2} \times \frac{1}{6}\right) + \left(\frac{1}{4} \times \frac{1}{2} \times \frac{5}{6}\right) = \frac{1}{12} + \frac{1}{48} + \frac{5}{48} = \frac{10}{48}$$

Alternatively, we could calculate this as $Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$, where Pr[E] the sum of 2 path probabilities, Pr[F] is the sum of 2 path probabilities, and $Pr[E \cap F]$ is the path probability for the one path which is in both E and F.

(Answer \mathbf{D})

Question 10

For the stochastic process modelled by the given tree, what is $\Pr[B|A]$?

Pr[B|A] is the probability shown on the tree, on the *B* branch that grows from the end of the *A* branch. (Answer **E**)

For the stochastic process modelled by the given tree, what is $\Pr[B \cap A^c]$?

This probability is just the path probability for the path which contains both A^c and B branches. We find the path probability by finding the product of the probabilities on the branches in the path, so we get $Pr[B \cap A^c] = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$. (Answer **B**)

Question 12

For the stochastic process modelled by the given tree, what is $\Pr[A|B]$

We use Bayes' theorem to get

$$Pr[A|B] = \frac{\text{path probability for path with both A and B branches}}{\text{sum of path probabilities for all paths with a B branch}}$$
$$= \frac{\frac{1}{4} \times \frac{1}{2}}{\left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{3}{4} \times \frac{1}{3}\right)}$$
$$= \frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{4}}$$
$$= \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}$$

(Answer \mathbf{C})

Question 13

A widget produced by a particular process has probability .1 of being defective. A test can be performed which has 99% accuracy. That is, if a defective widget is tested, the test will identify the widget as defective 99% of the time. And if a non-defective widget is tested, there is a 99% chance that the test will indicate that the widget is not defective. One widget is selected at random and is tested. If the test says that the widget is not defective, what is the probability that this widget actually is defective?

Let D be the event that a widget is defective and TD be the event that the test identifies a widget as being defective. We are told that Pr[D] = .1 and that $Pr[TD|D] = Pr[TD^C|D^C] = .99$ as well. We can draw a probability tree to model this process:



We are asked to find $Pr[D|TD^{C}]$. We use Bayes' Theorem.

$$Pr[D|TD^{C}] = \frac{\text{path probability for path with both } D \text{ and } TD^{C} \text{ branches}}{\text{sum of path probabilities for all paths with a } TD^{C} \text{ branch}}$$
$$= \frac{(.1)(.01)}{(.1)(.01) + (.9)(.99)}$$
$$= \frac{.001}{.001 + .891}$$
$$= \frac{.001}{.892}$$

(Answer E)

Question 14

A fair coin is tossed 4 times. What is the probability that heads will come up on the fourth toss, if it is known that heads has come up on **all** of the first 3 tosses?

No matter what has happened on the first 3 tosses, the probability that heads will come up on the next toss is $\frac{1}{2}$.

Note: Successive coin tosses are, as we know, independent. (Answer**B**)

Question 15

A coin is tossed 10 times. What is the probability that heads comes up exactly 5 times?

We are repeating the experiment 'toss a coin' 10 times. Since successive coin tosses are independent, we are performing n = 10 Bernoulli trials. Defining success to be that heads comes up, we have $p = \frac{1}{2}$. We want the probability of observing exactly k = 5 successes, which is given by

$$Pr[\text{exactly } k \text{ successes}] = \binom{n}{k} p^k (1-p)^{n-k} = \binom{10}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 = \binom{10}{5} \left(\frac{1}{2}\right)^{10}$$

 $(Answer \mathbf{D})$

Question 16

A die is rolled 5 times. What is the probability that one 2, two 3's and two 6's are rolled?

Repeated die rolls are also independent, so again we are performing independent trials. This time, there is more than one event of interest. Let E_1 be the event that a 2 is rolled, E_2 be the event that a 3 is rolled, E_3 be the event that a 6 is rolled and E_4 be the event that none of these are rolled (i.e. that a 1, 4 or 5 is rolled). Then we have $p_1 = p_2 = p_3 = \frac{1}{6}$ and $p_4 = \frac{1}{2}$. We want the probability that in n = 5 trials, E_1 occurs $n_1 = 1$ time, E_2 occurs $n_2 = 2$ times and E_3

occurs $n_3 = 2$ times, so that E_4 occurs $n_4 = 0$ times. This probability is given by

$$\binom{n}{n_1 n_2 n_3 n_4} (p_1)^{n_1} (p_2)^{n_2} (p_3)^{n_3} (p_4)^{n_4} = \binom{5}{1 \ 2 \ 2 \ 0} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{2}\right)^0$$
$$= \frac{5!}{1! \ 2! \ 2! \ 0!} \left(\frac{1}{6}\right)^5 \left(\frac{1}{2}\right)^0$$
$$= \frac{5!}{2! \ 2!} \left(\frac{1}{6}\right)^5$$

 $(Answer \mathbf{D})$

Question 17

A single card is drawn from a well-shuffled deck. This experiment is repeated 5 times. (The card is replaced and the deck is shuffled between repetitions of the experiment.) What is the probability that exactly one heart is drawn?

Since the card is replaced and the deck reshuffled between draws, we are sampling with replacement and these are independent trials. If we define success to be that a heart is drawn, we are looking for the probability of observing exactly k = 1 success in n = 5 Bernoulli trials with probability of success p = 1/4. We get

$$Pr[\text{exactly 1 success}] = {5 \choose 1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^4$$

 $(Answer \mathbf{C})$

Question 18

Refer to the series of experiments described in Question 17. What is the probability that at least one club is drawn?

If we do not draw at least one club, then we must draw no clubs. Thus, drawing at least one club is the complement of drawing no clubs. This time we define success to be that a club is drawn, which again gives p = 1/4. We have

$$Pr[\text{no successes}] = {\binom{5}{0}} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5 = 1 \times 1 \times \left(\frac{3}{4}\right)^5 = \left(\frac{3}{4}\right)^5$$

Therefore we see that

$$Pr[\text{at least one success}] = 1 - \left(\frac{3}{4}\right)^5$$

(Answer \mathbf{C})

Refer to the series of experiments described in Question 17. What is the probability that exactly 2 spades and exactly one diamond are drawn?

This time, as in question 16, there is more than one event of interest. Let E_1 be the event that a spade is drawn, E_2 be the event that a diamond is drawn and E_3 be the event that neither a spade nor a diamond is drawn (i.e. either a heart or a club is drawn). Then we have $p_1 = p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{2}$. We want the probability that in n = 5 trials, E_1 occurs $n_1 = 2$ times and E_2 occurs $n_2 = 1$ time. Of course, since there are 5 trials, and this only accounts for 3 of them, then event E_3 must occur $n_3 = 2$ times. We find the probability of this combination of outcomes to be

$$\binom{n}{n_1 n_2 n_3} (p_1)^{n_1} (p_2)^{n_2} (p_3)^{n_3} = \binom{5}{2 \ 1 \ 2} \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^1 \left(\frac{1}{2}\right)^2$$

 $(Answer \mathbf{A})$

Question 20

What is the smallest number of times you would need to toss a fair coin in order to be sure of having probability at least .9 of tossing at least one head?

Tossing a fair coin corresponds to performing Bernoulli trials. Let success be defined as heads being tossed, so that p = 1/2. As in question 18, the probability of observing at least one success is given by 1 - Pr[no successes], so when the coin is tossed *n* times, the probability that at least one head is tossed is given by $1 - \binom{n}{0} \left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^n$. We need this probability to be at least .9, so we need

$$1 - \left(\frac{1}{2}\right)^n \ge .9$$

$$\Rightarrow \quad 1 - .9 \ge \left(\frac{1}{2}\right)^n$$

$$\Rightarrow \quad .1 \ge \left(\frac{1}{2}\right)^n$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n \le .1$$

Using either trial and error or logarithms, we find that the smallest integer value of n for which this is true is n = 4. (Answer **C**)