Test 1 Solutions – Summer Evening 2003

Question 1

Which expression describes the set of all undergrads who are in Arts and either live in residence or own their own vehicle?

We want the set containing all student who are in set A *and* are in either set R or set V, i.e. are in at least one of set R or V, so in set notation this is the subset $A \cap (R \cup V)$.
(Answer **D**)

Question 2

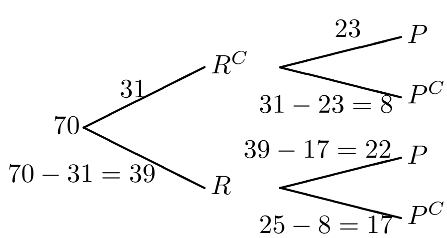
Which describes the set $A^C \cup (R \cap V)^C$?

This is the set of all students who are not in Arts or else are not both in R and in V, i.e. do not both live in residence and own their own vehicle. The only students who are not in this set are the students who *are* in Arts and *do* both live in residence and own their own vehicles. Therefore this set contains everyone *except for* the Arts students who live in residence and own a vehicle. (That is, $A^C \cup (R \cap V)^C = [A \cap (R \cap V)]^C$.)
(Answer **E**)

Question 3

There are 70 students registered in a certain online course this summer. Of these, only 31 have never taken the course before. If 25 students failed the midterm exam, and 23 of the students taking the course for the first time passed the midterm, how many of the students who are repeating the course failed the midterm? (Assume all registered students wrote the midterm.)

We can find this most easily with a counting tree to sort the students in the class according to whether each is repeating the course (R), and whether each passed the midterm (P):



We see that 17 of the students who are repeating the course failed the midterm.
(Answer **D**)

Question 4

There are 27 houses on Short Street. Of these, there are 13 houses which have a garage and do not have a covered porch. In total, there are 21 houses which either do not have a garage or do not have a covered porch. How many of the houses on Short Street have garages?

Letting G be the set of all houses which have garages and P be the set of all houses which have covered porches, we are told that $n(G \cap P^C) = 13$ and that $n(G^C \cup P^C) = 21$. Of course, $G^C \cup P^C = (G \cap P)^C$, and since $n(U) = 27$, we see that $n(G \cap P) = 27 - 21 = 6$. So we get $n(G) = n(G \cap P) + n(G \cap P^C) = 6 + 13 = 19$.
(Answer **B**)

Question 5

A certain car model comes in both 2-door and 4-door styles. Both styles are available in any of 5 different colours. Either style can be ordered with an optional hatch-back design. Taking into account the colour and style as well as whether or not the car has a hatchback or is a convertible, how many different ways are there in which to order a car of this model?

There are 2 separate cases to consider: the 2-door style and the 4-door style.
For the 2-door style, the car could be a hatch-back, or it could be a convertible, or it could be neither. (It cannot be both hatch-back and convertible.) This gives 3 designs, each of which is available in 5 colours, for a total of 15 different 2-door cars which could be ordered.
For the 4-door style, the car may or may not be a hatch-back. There are no other design choices available, so there are only 2 designs, each of which is available in any of the 5 colours, for a total of 10 4-door cars.
Therefore there are, altogether, $15 + 10 = 25$ different ways in which to order a car of this model.
(Answer**B**)

Question 6

The dessert table at a buffet has 3 kinds of pie, 4 different cakes, 3 fruit dishes and 5 flavours of ice cream. A customer at this buffet can sample as many of these different desserts as they wish. In how many ways can a customer decide which items to have for dessert?

A customer can have as few or as many different desserts as desired, so for each dessert item, the customer must decide 'do I want to have some of this?'. There are $3+4+3+5 = 15$ different dessert items available, so the customer is making 15 different decisions, each with 2 choices available (yes or no). This gives 2^{15} different ways to make the dessert decision.
(Answer **A**)

Question 7

John is a customer at the buffet and has decided to have 1 piece of pie, 1 sliver of cake, a scoop of the 'tropical fruit salad' and scoops of 2 different flavours of ice cream. In how many different ways can John select his dessert?

There are 3 pies to choose from, so there are $\binom{3}{1} = 3$ ways to choose one piece of pie. There are 4 cakes, so there are $\binom{4}{1} = 4$ ways to choose one cake to have a sliver of. He has already decided to have a specific one of the fruit choices, so there's only one way to have a scoop of the 'tropical fruit salad'. Finally, there are 5 different ice creams, so there are $\binom{5}{2} = 10$ ways to choose 2 flavours of ice cream. Therefore (by the FCP) John still has $3 \times 4 \times 1 \times 10 = 120$ ways in which he can select his dessert.
(Answer **C**)

Question 8

There are 15 books to be placed on a bookshelf. In how many different ways could these books be arranged on the shelf?

There are 15! different ways in which 15 distinct objects can be arranged.
(Answer **B**)

Question 9

There are 15 books to be placed on a bookshelf. Nine are hardcover books, while 6 are paperback. In how many ways can the books be arranged on the shelf so that books of the same type are together on the shelf?

There are 9! ways to arrange the 9 hardcovers in a group, and 6! ways to arrange the 6 paperbacks in a group. Then there are 2! ways to arrange these groups, i.e. 2 ways to decide whether the hardcovers are placed on the shelf to the right or to the left of the paperbacks. Thus there are 9!6!2! ways in which the books can be arranged.
(Answer **D**)

Question 10

There are 15 books to be placed on a bookshelf. The collection of books includes 3 books in the "An Idiot's Guide to ..." series. In how many ways can the books be arranged on the shelf if the 3 "Idiot's" books must be together?

The 15 books consist of the 3 "Idiot's" books plus 12 other books. If we consider the 3 "Idiot's" books as a single object, then there are a total of 13 objects to be arranged on the shelf. As well, we must arrange the "Idiot's" books within their group. Thus there are 13!3! arrangements of the books on the shelf so that the 3 "Idiot's" books are all together.
(Answer **C**)

Question 11

There are 4 married couples at a dinner party. The 8 people are to be arranged around a table for dinner. How many different seating arrangements are possible if the only restriction is that Mrs. Smith cannot be seated next to Mrs. Jones?

If we had no restrictions, there would be $(8-1)! = 7! = 5040$ ways to arrange the 8 people in a circle. Of these, many have Mrs. Smith sitting next to Mrs. Jones. How many? If we consider Mrs. Smith and Mrs. Jones as a single object, then we have 7 objects to arrange in a circle, which can be done in $6! = 720$ different ways. Of course, we must also consider how the single object of Mrs. Smith and Mrs. Jones is arranged. Mrs. Smith could be either to the right or to the left of Mrs. Jones. (i.e. there are $2! = 2$ ways to arrange the 2 women, within the one object). So there are $120 \times 2 = 1440$ different seating arrangements in which Mrs. Smith and Mrs. Jones *are* sitting together, which means that in all of the other $5040 - 1440 = 3600$ arrangements, they are not.
(Answer **E**)

Question 12

How many different seating arrangements are possible if men and women must alternate around the table?

We have 4 men and 4 women, and we want to arrange them in a circle so that men and women alternate around the circle. If we have the men sit first, there are $3! = 6$ ways in which they could arrange themselves at the table. After that, there are $4! = 24$ possible orderings of the women, i.e. ways that the women could come in and arrange themselves in between the men. Thus there are $6 \times 24 = 144$ ways in which the circle can be formed.
(Another approach for this would be to arrange the men and women in a line so that they alternate, which can be done in $4!4! \times 2$ ways, and then have the 8 people sit around the circle in the same order, which gives $\frac{1}{8} \times 4!4! \times 2 = \frac{24 \times 24 \times 2}{8} = 144$ ways to form the circle.)
(Answer **A**)

Question 13

La Belle Boutique needs to stock up on bathing suits. Because it is relatively late in the season, it has been decided that only 5 of the 12 designs they usually carry should be restocked. In how many different ways can the manager decide which 5 designs to reorder and which 7 designs not to restock?

There are 12 different designs, and they which to choose 5 to restock. This can be done in $\binom{12}{5}$ different ways.
(Answer**C**)

Question 14
A certain board of directors consists of 15 people. Three subcommittees are to be formed: a Finance subcommittee of 5 people, a Programs subcommittee of 4 people and a Fundraising subcommittee of 6 people. Each board member must sit on exactly 1 subcommittee, and the Treasurer (Mr. Bucks) must be on the Finance subcommittee. In how many ways can the subcommittees be chosen?

We have 15 board members to divide into 3 distinct (i.e. distinguishable) groups. However we already know that Mr. Bucks is going to be on the Finance subcommittee, so we really only have 14 people to divide up, and the Finance Committee will only have 4 other members. That is, we need to divide up a set of 14 people into 2 distinguishable groups of size 4 (the Finance subcommittee and the Programs subcommittee) and a group of size 6 (the Fundraising subcommittee). This can be done in $\binom{14}{4\ 4\ 6}$ different ways.
(Answer**B**)

Question 15
How many distinct arrangements of the letters in the word ASSININE are there?

There are 8 letters in the word ASSININE, consisting of 1 A, 2 S's, 2 I's, 2 N's and one E. Arranging the letters of a word in which there is repetition of letters is a labelling problem, where we think of labelling the 8 positions of the word, using 1 A label, 2 S labels, 2 I labels, 2 N labels and 1 E label. Therefore the number of ways of doing this is

$$\binom{8}{1\ 2\ 2\ 2\ 1} = \frac{8!}{1!2!2!2!1!} = \frac{8!}{2^3} = \frac{8 \times 7!}{8} = 7!$$

(Answer **B**)

Question 16
A Grandmother has decided to leave her 8 distinct display plates to her 8 grandchildren, but has not decided how they should be distributed. In how many different ways could the distribution of the 8 plates among the grandchildren be specified in the woman's will?

For each plate, the grandmother must decide 'which one of my 8 grandchildren will I leave this plate to?' Thus she has 8 decisions to make, with 8 choices for each decision. Therefore she could distribute the plates in any of 8^8 distinct ways.
(Answer **A**)

Question 17

The grandmother also has 8 identical mugs which she will leave to her grandchildren. Again she has not yet decided on how they should be distributed. In how many different ways could the woman decide to leave these 8 mugs to her grandchildren.

Now we are considering the number of ways of distributing 8 *identical* objects among 8 people. That is, we have $k = 8$ and $r = 8$ and we use the Free Distributions formula. We see that there are

$$\binom{k+r-1}{k} = \binom{8+8-1}{8} = \binom{15}{8}$$

different ways in which the mugs could be distributed.
(Answer**D**)

Question 18

Consider the experiment: Draw 2 cards (without replacement) from a standard deck. Consider also the following sets:

$S_1 = \{$2 red cards drawn, 2 black cards drawn, 1 red card and 1 black card drawn$\}$
$S_2 = \{$no hearts drawn, exactly 1 heart drawn, 2 hearts drawn$\}$
$S_3 = \{$at least 1 heart drawn, at least 1 black card drawn$\}$

Which of these sets are possible sample spaces for the experiment?

Since 2 cards are being drawn, the only colour possibilities are to get 2 reds, 1 red (and hence also one black) or no reds (i.e. get 2 blacks). Therefore set S_1 is complete, containing mutually exclusive descriptions of the cards, and is a possible sample space for the experiment.
Likewise, the only possibilities when we consider hearts is to get no hearts, 1 heart or 2 hearts. So S_2 is another complete set of mutually exclusive descriptions of the 2 cards, and is also a possible sample space.
Consider now set S_3. 'At least 1 heart drawn' means either 1 or 2 hearts drawn. If there is at least one black card drawn (either 1 or 2 black cards) then there can be no more than one red cards (either 0 or 1 red cards). There is overlap between 'at least 1 heart drawn' and 'at least 1 black card drawn' because drawing 1 heart and 1 black card is an outcome which fits both descriptions. Also, the set is not complete because a hand of 2 diamonds does not fit either description. Therefore S_3 is not a possible sample space.
(Answer **D**)

Question 19

Consider the experiment 'toss a coin 3 times'. Let H represent heads being tossed and T represent tails being tossed. Consider the set

$$S = \{\text{at least one } H, \text{ more than one } T\}$$

Which of the given statements is true?

We have 3 tosses, so there could be 0, 1, 2 or 3 heads. All outcomes with 1, 2 or 3 heads tossed are included in 'at least one H'. Also, tossing 0 heads, and therefore tossing 3 tails, is included in 'more than one T'. Therefore all possible outcomes appear in this set and the set S is complete.
The elements of the set are not mutually exclusive, though, because any outcome in which exactly 2 tails are tossed, and therefore one head is tossed, is included in both 'more than one T' and 'at least one H'.
(Answer **B**)

Question 20

Consider the experiment 'Choose 1 student from the Math 028b Summer Evening class'. Let S be the set whose sample points are the names of the students in the class. Consider the following events defined on this sample space:

K is the event that the student is registered at King's College,
P is the event that the student is in Psychology,
M is the event that the student is male.

Let E be the event that the chosen student is not a male King's student who is not in Psychology. Which of the given statements is a correct statement of the event E?

Any student who *is* a male King's student who is not in Psychology is in the set $M \cap K \cap P^C$. Therefore any student who is not one of these is in the complement of this set. That is, the set of all students who are *not* male King's students who are not in Psychology is the set $(M \cap K \cap P^C)^C$.
(Answer **E**)