Math 1229 Study Sheet - Test 1

What are vectors?

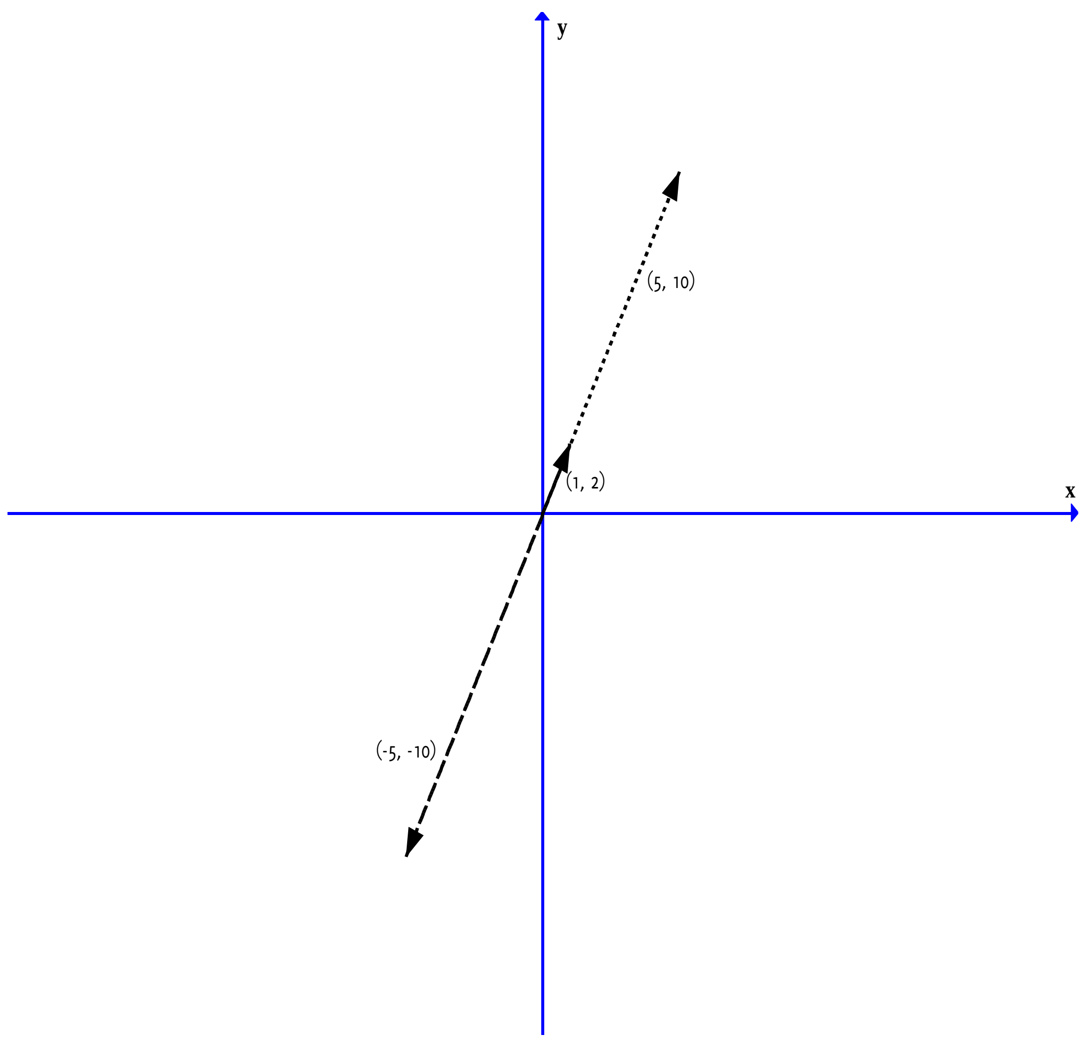
Vectors are things with *direction* and *length*. “12 miles in the north direction” represents a vector. “10 inches” does not represent a vector (since it doesn't have any direction). If you're trying to picture vectors, think of them as arrows. Sometimes it's better not to picture things (you might end up making the problem seem harder than it really is)

Adding and Multiplying Vectors

To add two vectors, make sure they have the same number of coordinates – you can't add (1, 2) and (2, 1, -3). Next, add the respective coordinates together: (1, 2) + (3, 4) = (4, 6). Vectors can be subtracted in the same way.

“Scalar” is just a fancy word for number.

To multiply a vector by a scalar, just multiply each coordinate by that number.
For example, 4(3, 1) = (12, 4)



If one vector is a multiple of another, then the vectors point in the same direction but have different lengths. (1, 2) and (5, 10) point in the same direction because (5, 10) = 5(1, 2) but (5, 10) will be 5 times as long as (1, 2). If a vector is a negative multiple of another, then it will point in the opposite direction. Since (-5, -10) = -5(1, 2), that means that (-5, -10) will be 5 times as long as (1, 2) and will point in the opposite direction.

Dot Product and Cross Product

To find the dot product of two vectors, simply multiply the respective coordinates together and then add the results. For example, (1, 2, 1) • (2, 0, -3) = 1X2 + 2X0 + 1X(-3) = 2 + 0 + (-3) = -1

To find the cross product of two vectors:

	Example: (1, 3, 2) x (4, 1, 0)
Step 1: Write the coordinates of the first vector horizontally twice	1 3 2 1 3 2
Step 2: Do the same to the second vector. Put this row directly below the first	1 3 2 1 3 2 4 1 0 4 1 0
Step 3: Eliminate the first and last columns	3 2 1 3 1 0 4 1
Step 4: Make an "X" between the first two columns. The first coordinate will be: 3X0 – 2X1 = 0 – 2 = -2	3 2 1 3 1 0 4 1 (1, 3, 2) x (4, 1, 0) = (-2, ?, ?)
Step 5: Move the "X" down a step and do the same thing. The second coordinate is: 2X4 – 0X1 = 8 – 0 = 8	3 2 1 3 1 0 4 1 (1, 3, 2) x (4, 1, 0) = (-2, 8, ?)
Step 6: Move the "X" down one last step and repeat. The third coordinate is: 1X1 – 3X4 = 1 – 12 = -11	3 2 1 3 1 0 4 1 (1, 3, 2) x (4, 1, 0) = (-2, 8, -11)

(This might sound complicated, but it's one of those things that seems harder when put into words. If you try it out, it's not that hard. And it's a lot easier than the method taught in the textbook or in class.)

Length and Unit Vectors

To find the length of a vector, square each coordinate and add them together. Then, take the square root of the result. The length is denoted by double vertical bars.

To find the length of (1, 4, -2):

$$\|(1,4,-2)\|$$
$$=\sqrt{1^2+4^2+(-2)^2}$$
$$=\sqrt{1+16+4}$$
$$=\sqrt{21}$$

(Length can also be called norm or magnitude)

A unit vector is a vector with length one. If you had vector and you wanted to find a unit vector pointing in the same direction, start by finding the length of that vector and then divide each coordinate by that length. (If you had a 10 foot pole and you wanted to scale it down to 1 foot, you'd divide it by 10. Likewise, if you had a $\sqrt{3}$ foot pole and you wanted to scale it down to 1 foot, you'd divide it by $\sqrt{3}$)

So, a unit vector pointing in the direction of (1, 4, -2) is:

$$\frac{1}{\sqrt{21}}(1,4,-2)$$

Parallel and Perpendicular Vectors

Two vectors are parallel if each of their coordinates are proportional – in other words, one vector should be a multiple of the other. For example, (1, 2, 1) and (4, 8, 4) are parallel because (4, 8, 4) = 4(1, 2, 1)

Two vectors are perpendicular if their dot product is equal to zero.

The dot product is used to test if two vectors are perpendicular. It always returns a number.
The dot product is useful for questions like "Find the value of k for which ... and ... are perpendicular"

The cross product is used to generate a new vector perpendicular to two given vectors. It always returns a vector. The cross product is useful for questions like "Find a vector perpendicular to ... and ..."

The dot product is also useful in questions that ask "Find the cosine of the angle between u and v" by applying the following formula:

$$\cos\theta=\frac{u\cdot v}{\|u\|\|v\|}$$

Area and Volume

The area of a triangle is

$$A = \frac{1}{2} \| u \times v \|$$

The area of a parallelogram is

$$A = \| u \times v \|$$

The volume of a parallelopiped (it's a three dimensional parallelogram – like a cube that got bent on its side) is

$$V = | u \cdot (v \times w) |$$

In the formulas above, u, v, w are the vectors that determine the parallelogram or parallelopiped. They'll be given to you in the question.

Equations of Lines

1. Point Parallel Form (or Point Vector Form)

To use this form, you need **a point (p)** on the line and **a vector (v)** pointing in the direction of the line (called a direction vector).

(x, y, z) = p + tv

So, if we wanted a line in the direction (1, 2, 1) going through (4, -1, 0), the point parallel form would be: (x, y, z) = (4, -1, 0) + t(1, 2, 1)

t is called a parameter. By plugging in different values for t, the equation will spit out different points on the line.

2. Point Normal Form

To use this form, you need a point (p) on the line and a vector perpendicular to the line. This vector is called a normal (n).

((x, y) – p) • n = 0

Point Normal Form gives a line if we're in R2. In higher dimensions, point normal form is used for planes.

3. Two Point Form

To use this form, you need two points on the line (p and q). The equation is:

(x, y, z) = tp + (1-t)q

For each of these equations, you can choose any point on the line or any direction vector. So, the equations won't be unique! In other words, different equations can actually represent the same line.

4. Parametric Equations

To find the parametric equations, you need a direction vector and a point on the line.
If your direction vector is (v_1, v_2, v_3) and your point is (p_1, p_2, p_3), then the parametric equations are:

$$x = p_1 + tv_1$$
$$y = p_2 + tv_2$$
$$z = p_3 + tv_3$$

These are very close to the point parallel form, except that each of x, y and z are listed separately

Equations of Planes

1. Point-Normal Form

Point normal form of a plane is the same as point normal form of a line, except we have three coordinates instead of just two. Again, you need a normal vector (n) and a point on the plane (p)

$$((x, y, z) - p) \bullet n = 0$$

2. Standard Form

$$n_1x + n_2y + n_3z = D$$

To put an equation in standard form, you need a normal vector and a point on the plane. The coordinates of the normal are going to be the coefficients of x, y and z in the equation.

If the normal is (4, 2, 3), then the standard form will be $4x + 2y + 3z = D$

D is equal to the dot product of the normal vector with a point on the plane. Alternatively, you can find D by substituting in the point for x, y and z and solving for D.

Distance from a Point to a Plane

To find the distance you'll need to know:
A point on the plane (we'll call it q)
A normal vector to the plane (call it n)
A point that you're trying to find the distance from (call it p)

Then the distance is:

$$\frac{|n \cdot (p - q)|}{\|n\|}$$

Other Important Stuff

- R^2 means 2 dimensional space. Vectors in R^2 will all have two coordinates. R^2 is like a flat surface. Likewise, R^3 means 3 dimensional space. Vectors in R^3 will all have three coordinates. R^3 is like the world we live in. We can also talk about R^4, R^5, etc., but it gets difficult to visualize them.

- You can't add vectors with a different number of coordinates. For example, (1, 2) + (1, 4, 2) doesn't make any sense. If it shows up on a test, the answer is "undefined." Or else, put up your hand and tell the prof there's a typo. In fact, you can't take the dot product or cross product of vectors with a different number of coordinates either.

- "Orthogonal" means the same as "perpendicular."

- A "scalar" is just a number (as opposed to a vector)

- If you are given a two coordinate vector, an easy way to find a normal to it is to reverse the coordinates and change one of the signs. So, a normal to (3, 2) would be (-2, 3) or (2, -3).

- To determine if a point lies on a line (or plane) just substitute the point into the equation and see if it makes sense. For example, if we want to find out if (1, 4) lies on the line $9x - 2y = 1$, let $x = 1$ and $y = 4$:

  $9(1) - 2(4) = 9 - 8 = 1$. So the equation is true and the point lies on the line.

  Likewise, to find out if (1, 3) is on that same line:

  $9(1) - 2(3) = 9 - 6 = 3$. But the equation says we should get 1. So, (1, 3) is not on the line.

To find the intersection of a plane and a line

1. Put the line into parametric form
2. Substitute the parametric equations into the standard form of the plane and solve for t
3. Substitute the solved value for t back into the original parametric equations of the line

Example:
Find the intersection of the line $(x, y, z) = (1, 2, 1) + t(1, 2, 3)$ and the plane $2x - y + 3z = 12$

1. Put the line into parametric form:
   $x = 1 + t$
   $y = 2 + 2t$
   $z = 1 + 3t$

2. Substitute the parametric equations into the standard form of the pane:
   $2x - y + 3z = 12$
   $2(1 + t) - (2 + 2t) + 3(1 + 3t) = 12$
   $2 + 2t - 2 - 2t + 3 + 9t = 12$
   $9t = 9$
   $t = 1$

3. Substitute the solved value for t back into the original parametric equations of the line
   $(x, y, z) = (1, 2, 1) + t(1, 2, 3)$
   $(x, y, z) = (1, 2, 1) + 1(1, 2, 3)$
   $(x, y, z) = (2, 4, 4)$

   So, the intersection point is (2, 4, 4)

Good Luck on your Test!