# PHAS1224 Waves, Optics and Acoustics Part I: Waves and Acoustics 

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## 1 Preliminaries

### 1.1 What is a wave?

The following quote comes from the book "The evolution of physics" by Einstein and Infeld:
"A bit of gossip starting in London reaches Edinburgh very quickly, even though not a single individual who takes part in spreading it travels between these two cities. There are two quite different motions involved, that of the rumour, London to Edinburgh, and that of the persons who spread the rumour. The wind, passing over a field of grain, sets up a wave which spreads our across the whole field. Here again we must distinguish between the motion of the wave and the motion of the separate plants, which undergo only small oscillations. We have all seen the waves that spread in wider and wider circles when a stone is thrown into a pool of water. The motion of the wave is very different from that of the particles of water. The particles merely go up and down...The particles constituting the medium perform only small vibrations, but the whole motion is that of a progressive wave. The essentially new thing here is that for the first time we consider the motion of something which is not matter, but energy propagated through matter."

This offers a good introductory idea of what we mean by a wave, though it is important to stress that electromagnetic waves will propagate without a medium. In fact, this is what led to the development of special relativity, and is extremely important in the history of physics. We will start by re-examining harmonic oscillations (as the disturbance which propagates as a wave will undergo harmonic motion at each point when we have a periodic wave). We will then derive a general wave equation and explore its applications to different types of wave.

### 1.2 Prerequisites

In order to take this course, students should be familiar with the basic principles of physics to a standard comparable with a grade A or B in GCSE Advanced Level, and to have a level of competence in mathematics consistent with having passed course PHAS1245/46.

### 1.3 Syllabus

The syllabus for the first part of the course is given below, with a rough allocation of lectures for each section.
Introduction[2]: Simple harmonic motion; phasors; complex number representation; beats
Basic Waves[2]: types of wave motion; progressive waves; simple harmonic form; definitions of amplitude, frequency etc.; phase and phase velocity; general differential equation of wave motion; superposition

Transverse waves[3]: stretched string; reflection and transmission at boundaries; impedance; energy/energy propagation; impedance matching; standing waves and normal modes

Longitudinal waves[1]: sound in gases; sound in solid rod
Doppler effect[1]
Dispersive Waves[2]: dispersion relations; phase velocity and group velocity

### 1.4 Aims and Objectives

Note that these aims and objectives apply to the whole course, though these notes are only for the first half of the course. This course aims to provide:

- an account of the phenomenon of wave propagation and the properties of the wave equation in general, in a form which can be applied to a range of physical phenomena;
- an explanation of the way in which wave equations arise in some specific cases (transverse waves on a string; longitudinal sound waves in gases and solids);
- a discussion of reflection and refraction, illustrating the relationship between the wave and geometric (ray) pictures;
- a description of phenomena which arise from the superposition of waves, in- cluding interference and diffraction;
- an overview of simple optical devices in terms of geometric optics, including small numbers of lenses and curved mirrors, with a discussion of the limitations placed on such devices by diffraction;
- an introduction to simple optical devices which rely on interference;
- a description of the propagation of waves in free space and in simple enclosures;and,
- a foundation for the description of quantum mechanical wave phenomena in course PHAS2222 Quantum Physics and the theory of electromagnetic waves in Electromagnetic Theory.
After completing this half-unit course, the student should be able to:
- discuss the relationship between simple harmonic motion and wave motion, and make calculations of the motion of systems of masses connected by springs;
- make calculations on simple properties of wave motion, including wave packets, phase velocity, group velocity, and the propagation of waves in one, two and three dimensions;
- use the complex exponential representation for waves, and extract real values from it for the description of observable quantities;
- discuss the propagation of energy in waves;
- make calculations on and draw graphs illustrating the superposition of waves
- phenomenon of beats;
- make calculations on systems with moving sources and receivers (the Doppler effect).
- sketch and describe standing waves, especially on strings and in pipes with various boundary conditions, and make calculations on them;
- draw phasor diagrams for systems of interfering waves and use these or complex number methods to derive general formulae and to describe specific cases;
- derive the wave equation for transverse waves on a stretched string, and for longitudinal waves in compressible materials;
- describe polarization of transverse waves, and discuss the phenomena which arise therefrom;
- derive formulae for the reflection and transmission coefficients of waves at barriers, express them in terms of impedances, and apply them;
- describe and make calculations on simple guided wave systems;
- describe what is meant by phase coherence, and explain the qualitative difference between light from different types of source;
- describe Huygenss principle and apply it to simple cases;
- derive formulae for the diffraction patterns of single and double slits, and for diffraction gratings with narrow and finite slits;
- derive the criterion for the resolving power of a grating;
- draw geometric ray diagrams for simple systems involving prisms, lenses, and
- d curved reflecting surfaces;
- make calculations in geometric optics for simple systems involving slabs, prisms, lenses, and plane and curved reflecting surfaces;
- describe the operation of simple optical instruments, derive object and image positions and magnifications, and discuss quantitatively their resolving power;
- calculate the properties of and describe applications of the Michelson and Fabry-Perot interferometers.


### 1.5 Textbooks

Most of the course material is covered in the basic First Year textbook: Physics for Scientists and Engineers with Modern Physics by Jewett \& Serway (Brooks/Cole) though any first year level text would cover the material adequately; I give references to the appropriate chapters in Jerway \& Sewett for different sections.

As well as this overview, there are a number of excellent textbooks on waves and vibrations, such as:

- Iain G. Main, The Physics of Vibrations and Waves
- A. P. French, Vibrations and Waves
- H. J. Pain, The Physics of Vibrations and Waves

You may well find that one of these books suits your way of working and gives you a deeper understanding of the course material as well as more worked examples and problems to practise.

### 1.6 Assessment

You will be set four problem sheets during the course (two on each part of the course). These will not be marked, but worked solutions will be posted one week after the problem sheet. You will be assessed via a short ( 50 minute) test in an In-Class Assessment, which will consist of problems drawn at random from the problem sheets already set. This, along with a similar test based on the second half of the course, will form $15 \%$ of your final mark, with the remaining $85 \%$ coming from the examination in the summer. You should attempt to do all problem sheets, not only as they count towards your exam mark, but also as they will help you understand the course material and prepare you for the exam.

## 2 Simple Harmonic Motion (J\&S 15)

We will start by reviewing simple harmonic motion (SHM) as it contains many of the important concepts that we will meet in waves. The most general form of the equation for a simple harmonic oscillator (SHO) including damping and driving forces can be written:

$$
\begin{equation*}
m \frac{\partial^{2} \psi}{\partial t^{2}}=-s \psi-b \frac{\partial \psi}{\partial t}+F_{0} \cos \omega t \tag{1}
\end{equation*}
$$

where $\psi$ is the displacement, $m$ is the mass of the oscillator, $s$ is the constant giving the restoring force (e.g. a spring constant) sometimes known as the stiffness, $b$ is the damping coefficient or resistance and $F_{0}$ gives the magnitude of the driving force (which has angular frequency $\omega$ ).

### 2.1 SHM and Circular Motion (J\&S 15.1-15.4)

If we set the damping and driving coefficients to zero, we recover the original, SHM equation:

$$
\begin{equation*}
m \frac{\partial^{2} \psi}{\partial t^{2}}=-s \psi \tag{2}
\end{equation*}
$$

which can be shown to be solved using sinusoidal motion:

$$
\begin{equation*}
\psi=A \sin \omega_{0} t+B \cos \omega_{0} t=C \cos (\omega t+\phi), \tag{3}
\end{equation*}
$$

where $\omega_{0}=\sqrt{s / m}, \phi$ is a constant phase and $A=-C \sin \phi$ and $B=C \cos \phi$. As you have seen in both PHAS1245 (Mathematical Methods I) and PHAS1247 (Classical Mechanics), we can use De Moivre's theorem to write the sinusoidal terms as a complex exponential ( $e^{i \theta}=\cos \theta+i \sin \theta$ where $i=\sqrt{-1}$ ), giving:

$$
\begin{equation*}
\psi=\operatorname{Re}\left[A e^{i\left(\omega_{0} t+\phi\right)}\right]=\operatorname{Re}\left[D e^{i \omega_{0} t}\right] \tag{4}
\end{equation*}
$$

where $D=A e^{i \phi}$.
We often call the argument of a trigonometric function the phase, and $\phi$ is often called the phase difference when there is more than one oscillation; it represents the offset in the phase at $t=0$. Now that we have the representation of the oscillation in terms of the complex exponential (or the sum of a cos and $\sin$ ) we can see that there is an immediate link with circular motion: the amplitude of the oscillation is just the projection of circular motion onto the $x$-axis (or any other axis that is chosen). (Note that from now on I will not write the need to take the real part of $\psi$ explicitly, but will assume it.) All these ideas are illustrated in Fig. 11a).


Figure 1: (a) Relation between circular motion of the vector $\psi$ and simple harmonic motion. (b) The phase relations between displacement (i), velocity (ii) and acceleration (iii) for a simple harmonic oscillator. Note that the amplitudes are $A, \omega A$ and $\omega^{2} A$ and are not to scale.

The representation of a simple harmonic oscillator at a single point in time in the complex plane (e.g. Fig. 1 (a)) is often called a phasor diagram, and the arrow representing the amplitude and phase of the oscillator relative to a fixed time
or phase is called a phasor; note that for a phasor we just need the arrow, not the associated circle. We will see later that we can combine two oscillations using phasors (or using complex arithmetic - the two are completely equivalent).

A simple way to think about phase differences is to consider the velocity and acceleration of the oscillator:

$$
\begin{align*}
\psi & =A e^{i\left(\omega_{0} t+\phi\right)}  \tag{5}\\
\dot{\psi} & =\frac{\partial \psi}{\partial t}=i \omega A e^{i\left(\omega_{0} t+\phi\right)}=i \omega \psi  \tag{6}\\
\ddot{\psi} & =\frac{\partial^{2} \psi}{\partial t^{2}}=-\omega^{2} A e^{i\left(\omega_{0} t+\phi\right)}=-\omega^{2} \psi \tag{7}
\end{align*}
$$

Notice that both of these quantities vary harmonically and with the same frequency as the oscillation, though with different amplitudes; more importantly, however, note that the velocity is a factor of $i$ different to the displacement, while the acceleration is a factor of -1 different. These are easier to understand when we write them in complex exponential form: $i=e^{i \pi / 2}$ and $-1=e^{i \pi}$. So we can write:

$$
\begin{align*}
\psi & =A e^{i\left(\omega_{0} t+\phi\right)}  \tag{8}\\
\dot{\psi} & =\omega A e^{i\left(\omega_{0} t+\phi+\pi / 2\right)}=\omega \psi e^{i \pi / 2}  \tag{9}\\
\ddot{\psi} & =\omega^{2} A e^{i\left(\omega_{0} t+\phi+\pi\right)}=\omega^{2} \psi e^{i \pi} \tag{10}
\end{align*}
$$

We see that the velocity leads the displacement by a phase factor of $\pi / 2$ and the acceleration leads the velocity by a phase factor of $\pi / 2$. This phase relationship is shown in Fig. 11 b).

The simple harmonic oscillator, as with all mechanical systems, has two forms of energy: kinetic and potential. If we have $W$ as the total energy and $T$ as kinetic and $V$ as potential, we can write:

$$
\begin{align*}
T & =\frac{1}{2} m \dot{\psi}^{2}  \tag{11}\\
V & =\frac{1}{2} s \psi^{2}  \tag{12}\\
W & =T+V \tag{13}
\end{align*}
$$

The form of the potential energy follows directly from the form of the force (and is why the motion is called harmonic). As there is no form of dissipation in the motion, we can assert that the total energy does not change with time, just exchanging between potential (at a maximum when the displacement is at a maximum and the velocity is at a minimum) and kinetic (at a maximum when the displacement is at a minimum). They are out of phase - as we expect from the phase differences between displacement and velocity.

As the total energy is constant, we can write:

$$
\begin{align*}
\frac{d W}{d t}=\frac{d T}{d t}+\frac{d V}{d t} & =0  \tag{14}\\
\Rightarrow m \dot{\psi} \ddot{\psi}+s \psi \dot{\psi} & =0  \tag{15}\\
\Rightarrow m \ddot{\psi} & =-s \psi \tag{16}
\end{align*}
$$

which is of course just the original equation for simple harmonic motion.

### 2.2 Damping oscillations (J\&S 15.6)

Restoring the damping term changes the equation and its solution. We will use $\gamma=b / 2 m$ in the equations below. These solutions were derived in PHAS1247 and are discussed in the textbook, so I will not derive them here. We find:

$$
\begin{align*}
m \frac{\partial^{2} \psi}{\partial t^{2}} & =-s \psi-b \frac{\partial \psi}{\partial t}  \tag{17}\\
\psi & = \begin{cases}A e^{-\gamma t} e^{i \omega t} & \gamma<\omega_{0} \\
\omega=\sqrt{\omega_{0}^{2}-\gamma^{2}} & \\
A e^{-\mu_{+} t}+B e^{-\mu_{-} t} & \gamma>\omega_{0} \\
\mu_{ \pm}=\gamma \mp \sqrt{\gamma^{2}-\omega_{0}^{2}} & \\
A\left(1+\omega_{0} t\right) e^{-\omega_{0} t} & \gamma=\omega_{0}\end{cases} \tag{18}
\end{align*}
$$

The cases for damped harmonic motion correspond to light (or underdamped, $\gamma<\omega_{0}$ ), heavy (or overdamped, $\gamma>\omega_{0}$ ) and critical ( $\gamma=\omega_{0}$ ) damping respectively. The total energy is not conserved in this case (contrast the undamped oscillator) as the damping force opposes the motion at all times. We write the energy as before, and find the change with time:

$$
\begin{align*}
W & =\frac{1}{2} m \dot{\psi}^{2}+\frac{1}{2} s \psi^{2}  \tag{19}\\
\frac{d W}{d t} & =\frac{d W}{d \dot{\psi}} \frac{d \dot{\psi}}{d t}+\frac{d W}{d \psi} \frac{d \psi}{d t}=(m \ddot{\psi}+s \psi) \dot{\psi}  \tag{20}\\
& =-b \dot{\psi}^{2} \tag{21}
\end{align*}
$$

where the last line comes from using Eq. 17). Notice that the change in energy is always less than zero (unless $\dot{\psi}=0$ ) and so the total energy of the system decreases with time (as we would expect).

### 2.3 Driving oscillations (J\&S 15.7)

Now we will introduce a driving term to the system. As we have a harmonic oscillator, it will make sense to use a harmonic driving term (though we could, for instance, use impulses at regular or irregular intervals). Note that the driving term will have an angular frequency $\omega$ which is different to the natural frequency of the system, $\omega_{0}=\sqrt{s / m}$. Also remember that $\omega=2 \pi \nu$ where $\nu$ is the frequency. We will retain the damping term to give a damped, driven oscillator (again derived in PHAS1247 and the textbook).

$$
\begin{align*}
m \frac{\partial^{2} \psi}{\partial t^{2}} & =-s \psi-b \frac{\partial \psi}{\partial t}+F_{0} \cos \omega t  \tag{22}\\
\psi & =A \cos (\omega t+\phi)  \tag{23}\\
A & =\frac{F_{0}}{m}\left(\frac{1}{\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+4 \gamma^{2} \omega^{2}}\right)^{\frac{1}{2}}  \tag{24}\\
\tan \phi & =\frac{-2 \gamma \omega}{\omega_{0}^{2}-\omega^{2}} \tag{25}
\end{align*}
$$

The response of the system (including both the phase and the amplitude) depends strongly on the frequency; we can consider three regimes (though there isn't really time to do this in depth).

1. When $\omega$ is small (i.e. $\omega \ll \omega_{0}$ ), the amplitude can be shown to be $A \simeq F_{0} / m \omega_{0}^{2}=F_{0} / s$ and the motion is dominated by the spring constant (stiffness controlled).
2. When $\omega$ is large (i.e. $\omega \gg \omega_{0}$ ), then the amplitude is $A \simeq F_{0} / m \omega^{2}$ and the motion is dominated by the mass (mass controlled).
3. When $\omega \sim \omega_{0}$, we are at resonance, and the response is controlled by the drag term $\gamma$ (also known resistance limited).

Power absorption Let's think a little about the power absorbed by the oscillator; to do that, we must consider the work done against the drag. If the displacement changes from $\psi$ to $\psi+\Delta \psi$ then the work done is $-F_{d} \Delta \psi$, where $F_{d}=-b \dot{\psi}$ is the work done against the drag. If that takes a time $\Delta t$ then the rate of work is $-F_{d}(\Delta \psi / \Delta t)$ which tends to $-F_{d} \dot{\psi}$ as $\Delta t \rightarrow 0$. So the instantaneous power adsorption becomes:

$$
\begin{equation*}
P=-F_{d} \dot{\psi}=b \dot{\psi}^{2} \tag{26}
\end{equation*}
$$

We are not actually interested in the instantaneous power, but the time averaged power (often written $\langle P\rangle$ ), which for a harmonic force can be shown to b ${ }^{1}$.

$$
\begin{align*}
\langle P\rangle & =b\left\langle\dot{\psi}^{2}\right\rangle=\frac{1}{2} b \omega^{2} A^{2}  \tag{27}\\
& =\frac{F_{0}^{2}}{m^{2}} \frac{m \gamma \omega^{2}}{\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+4 \gamma^{2} \omega^{2}} \tag{28}
\end{align*}
$$

where we have substituted $b=2 m \gamma$. This has a maximum value of $F_{0}^{2} /(4 m \gamma)$ when $\omega=\omega_{0}$. Note that this is inversely proportional to the resistive force.

Finally we introduce the idea of impedance which is a measure of the resistance to the motion of the oscillator. It is defined as the amplitude of the driving force divided by the complex amplitude of the velocity, $Z(\omega)=F_{0} / \dot{\psi}=$ $b+i(m \omega-s / \omega)$. At resonance, $Z(\omega)=b$; in the resonance region, it only departs very slightly from this value. Away from resonance, it includes a phase lag or lead, which reflects the relation of velocity to the driving force. We will encounter impedance again with waves.

[^0]Transients Note that this is a steady state solution: there is a transient behaviour at the beginning of the oscillation which takes the form of the solution in Eq. (23) added to a solution of the homogenous equation (i.e. the equation with no driving term-Eq. (17) or earlier). So a real system will respond at its natural frequency, but damping will cause that solution to die off, while the driven oscillation persists into the steady state:

$$
\begin{equation*}
\psi(t)=A_{s} \cos (\omega t+\phi)+A_{t} e^{-\gamma t} \cos \left(\omega_{0} t+\phi_{t}\right) \tag{29}
\end{equation*}
$$

where the subscripts s and t stand for steady state and transient respectively. For lightly damped system near resonance we can approximate and find that $A_{t} \simeq A_{s}$ and $\phi_{t} \simeq \phi_{s}+\pi$, giving:

$$
\begin{equation*}
\psi(t)=A_{s}\left(\cos (\omega t+\phi)-e^{-\gamma t} \cos \left(\omega_{0} t+\phi_{s}\right)\right) \tag{30}
\end{equation*}
$$

We have used the principle of superposition in finding this solution: as the oscillator equation is linear in $\psi$ we can always create a new solution for the wave equation as a linear combination of existing solutions. (If the basic equation was $\partial^{2} \psi / \partial t^{2}=-\omega_{0}^{2} \psi^{2}$ say then this would not be possible.) If we switch off the driving force, then we will again have transient behaviour-which will be just a damped harmonic oscillator. Think of a child on a swing: the parent pushing provides the driving force, which is switched off when the parent gets tired. Fortunately for all concerned, the swing comes to rest after some time (though we expect that the swing will be lightly damped).

We have introduced two important concepts here:

- Superposition of solutions giving a resultant shape to the oscillations
- The idea that, in all real physical processes, there is a beginning, a steady state (the middle if you like) and an end.

These ideas will both come back when we study waves.

### 2.4 Combining oscillations (J\&S 18.7)

Instead of combining a single driving force with a damped response, what happens if we have an oscillator with two driving forces? The full equation looks like this:

$$
\begin{equation*}
m \frac{\partial^{2} \psi}{\partial t^{2}}=-s \psi-b \frac{\partial \psi}{\partial t}+F_{1} \cos \left(\omega_{1} t+\phi_{1}\right)+F_{2} \cos \left(\omega_{2} t+\phi_{2}\right), \tag{31}
\end{equation*}
$$

We know that the SHO equation is linear, so we can try solving this problem by summing the steady state solutions arising from each driving force independently. We will consider three specific cases (where it is easier to understand the behaviour) with the general case more complex:

1. Same amplitude and frequency $\left(F_{1}=F_{2}=F, \omega_{1}=\omega_{2}=\omega\right)$, different phase ( $\phi_{1} \neq \phi_{2}$ )
2. Same frequency $\left(\omega_{1}=\omega_{2}\right)$, different phase and amplitude ( $F_{1} \neq F_{2}, \phi_{1} \neq \phi_{2}$ )
3. Same amplitude and phase $\left(F_{1}=F_{2}=F, \phi_{1}=\phi_{2}\right)$, different frequency $\left(\omega_{1} \neq \omega_{2}\right)$

In all cases, we assume that we can write the displacement as a superposition of the responses to the individual oscillations:

$$
\begin{equation*}
\psi=A_{1} \cos \left(\omega_{1} t+\phi_{1}\right)+A_{2} \cos \left(\omega_{2} t+\phi_{2}\right) \tag{32}
\end{equation*}
$$

Same amplitude and frequency For simplicity, we will set $\phi_{1}=0$ and set $\phi_{2}=\phi$. Then we can perform the following manipulations, using the complex exponential form for simplicity:

$$
\begin{align*}
\psi & =A e^{i \omega t}+A e^{i(\omega t+\phi)}  \tag{33}\\
& =A e^{i \omega t}\left(1+e^{i \phi}\right)  \tag{34}\\
& =A e^{i \omega t}\left(e^{i \phi / 2} e^{-i \phi / 2}+e^{i \phi / 2} e^{i \phi / 2}\right)  \tag{35}\\
& =A e^{i \omega t} e^{i \phi / 2}\left(e^{-i \phi / 2}+e^{i \phi / 2}\right)  \tag{36}\\
& =2 A e^{i(\omega t+\phi / 2)} \cos (\phi / 2)=2 A \cos (\phi / 2) e^{i(\omega t+\phi / 2)} \tag{37}
\end{align*}
$$

where the identification of $\cos (\phi / 2)$ follows from De Moivre's theorem. So the resultant oscillation has magnitude $2 A \cos (\phi / 2)$ and phase $\phi / 2$. The same result can be found using trigonometry and a phasor diagram, as illustrated in Fig. 2 (a).


Figure 2: (a) Phasors with the same frequency and amplitude. (b) Phasors with same frequencies. The resultant phasor A is made from the two driving phasors with amplitudes $\mathrm{A}_{1}$ and $\mathrm{A}_{2}$ and phases $\phi_{1}$ and $\phi_{2}$. (c) Phasors with different frequencies and amplitudes (see Fig. 4 for path of tip).

Same frequency We can write a solution as before, by using superposition:

$$
\begin{equation*}
\psi=A_{1} \cos \left(\omega t+\phi_{1}\right)+A_{2} \cos \left(\omega t+\phi_{2}\right)=A \cos (\omega t+\theta) \tag{38}
\end{equation*}
$$

This is illustrated in a phasor diagram in Fig. 2 (b). It is just an oscillation with frequency $\omega$ but a total amplitude and phase which depend on the amplitudes and phases of the two driving forces. If, for instance, the phase difference is $\phi_{1}-\phi_{2}=\pi$ then the two driving forces are out-of-phase and we will get destructive interference. The use of phasors allows a simple visualisation of the resultant. Using trigonometry from the phasor diagram (or the cosine rule), we can write the amplitude of the resultant as:

$$
\begin{equation*}
A^{2}=A_{1}^{2}+A_{2}^{2}+2 A_{1} A_{2} \cos \left(\phi_{2}-\phi_{1}\right) \tag{39}
\end{equation*}
$$

Now we can see that the amplitude will vary between $A_{1}+A_{2}$ if $\phi_{1}=\phi_{2}$ and $\left|A_{1}-A_{2}\right|$ if $\left|\phi_{1}-\phi_{2}\right|=\pi$. If the driving forces are in phase then we have the maximum amplitude, while if they are in antiphase we have the minimum amplitude.

The phase is also found using trigonometry; by projecting onto the real and imaginary axes, we can write:

$$
\begin{equation*}
\tan \theta=\frac{A_{1} \sin \phi_{1}+A_{2} \sin \phi_{2}}{A_{1} \cos \phi_{1}+A_{2} \cos \phi_{2}} \tag{40}
\end{equation*}
$$

We can get the same result using complex notation. Start by finding that, for two general complex numbers $z_{1}$ and $z_{2}$ :

$$
\begin{align*}
\left|z_{1}+z_{2}\right|^{2} & =\left(z_{1}+z_{2}\right)\left(z_{1}+z_{2}\right)^{\star}=\left(z_{1}+z_{2}\right)\left(z_{1}^{\star}+z_{2}^{\star}\right)  \tag{41}\\
& =\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}+\left(z_{1} z_{2}^{\star}+z_{1}^{\star} z_{2}\right)  \tag{42}\\
& =\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}+2 \operatorname{Re}\left(z_{1} z_{2}^{\star}\right) \tag{43}
\end{align*}
$$

Now we have $z_{1}=A_{1} e^{i\left(\omega t+\phi_{1}\right)}$ and $z_{2}=A_{2} e^{i\left(\omega t+\phi_{2}\right)}$ with $A_{1}$ and $A_{2}$ real. This gives:

$$
\begin{align*}
\left|z_{1}+z_{2}\right|^{2} & =\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}+2 \Re\left(z_{1} z_{2}^{\star}\right)  \tag{44}\\
& =A_{1}^{2}+A_{2}^{2}+2 \operatorname{Re}\left(A_{1} e^{i\left(\omega t+\phi_{1}\right)} A_{2} e^{-i\left(\omega t+\phi_{2}\right)}\right)  \tag{45}\\
& =A_{1}^{2}+A_{2}^{2}+2 \operatorname{Re}\left(A_{1} A_{2} e^{i\left(\phi_{1}-\phi_{2}\right)}\right)  \tag{46}\\
& =A_{1}^{2}+A_{2}^{2}+2 A_{1} A_{2} \cos \left(\phi_{2}-\phi_{1}\right) \tag{47}
\end{align*}
$$

while we can find the phase from:

$$
\begin{align*}
\arg \left(z_{1}+z_{2}\right) & =\tan ^{-1}\left[\frac{\operatorname{Im}\left(z_{1}+z_{2}\right)}{\operatorname{Re}\left(z_{1}+z_{2}\right)}\right]  \tag{48}\\
\Rightarrow \theta & =\tan ^{-1}\left[\frac{A_{1} \sin \phi_{1}+A_{2} \sin \phi_{2}}{A_{1} \cos \phi_{1}+A_{2} \cos \phi_{2}}\right] \tag{49}
\end{align*}
$$

Same amplitude and phase: beats For this case, we use the principle of superposition to write:

$$
\begin{equation*}
\psi=A \cos \omega_{1} t+A \cos \omega_{2} t \tag{50}
\end{equation*}
$$

Note that we have assumed that the common phase can be set to zero for simplicity. But we can rearrange this using a standard trigonometrical formula for the sum of two cosines:

$$
\begin{align*}
\psi & =A\left(\cos \omega_{1} t+\cos \omega_{2} t\right)  \tag{51}\\
& =2 A \cos \left(\frac{\omega_{1}+\omega_{2}}{2} t\right) \cos \left(\frac{\omega_{1}-\omega_{2}}{2} t\right) \tag{52}
\end{align*}
$$



Figure 3: (a) The result of two driving forces on one SHO with different frequencies but the same amplitude. Top: two forces shown together. Middle: the sum and difference oscillations. Bottom: the resultant motion with the envelope superimposed in dashed lines. (b) Two driving forces on one SHO (the frequencies do not have integer relationship). Top: first force. Middle: second force; note that the peaks and troughs of the two forces do not quite coincide. Bottom: the resultant motion with the envelope superimposed in dashed lines.

Now let's define two new angular frequencies, and rewrite the solution:

$$
\begin{align*}
\omega & =\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)  \tag{53}\\
\Delta \omega & =\frac{1}{2}\left(\omega_{1}-\omega_{2}\right)  \tag{54}\\
\psi & =2 A \cos (\omega t) \cos (\Delta \omega t) \tag{55}
\end{align*}
$$

If the two angular frequencies are relatively close, then the form of the resulting oscillation is rather simple. There is an oscillation at the average frequency (the term $\cos \omega t$ ) whose amplitude is modulated by a slow oscillation at the difference frequency (the term $\cos \Delta \omega t$ - also known as the envelope). This is a phenomenon known as beats, and is illustrated in Fig. 3 . It is important to understand that, while the angular frequency of the modulation is $\Delta \omega=\frac{1}{2}\left|\omega_{1}-\omega_{2}\right|$, the angular frequency at which peaks of activity occur is $\left|\omega_{1}-\omega_{2}\right|$ (or equivalently zeroes of activity). The number of minima per second is $\Delta \omega / \pi$. The perceived effect (say for sound) will be of a sound at the average angular frequency with its amplitude varying according to the envelope.

Figure 3(a) shows an illustration of exactly this behaviour for the frequencies $\omega_{1}=1.2 \mathrm{~s}^{-1}$ and $\omega_{2}=1.0 \mathrm{~s}^{-1}$ which gives a resulting motion at frequency $\omega=1.1 \mathrm{~s}^{-1}$ modulated by an envelope with frequency $\Delta \omega=0.1 \mathrm{~s}^{-1}$.

The resulting motion will show a true periodicity if the ratio of the frequencies can be written as a ratio of integers (i.e. $\omega_{1} / \omega_{2}=n_{1} / n_{2}$ ). The motion from two frequencies which are almost non-periodic is shown in Fig. 3(b) (I used $25 / 3$ and 7.1 here but we could have made it properly non-periodic if we'd tried harder).

Note that if there is a phase difference between the two oscillations then beats still result, but the phase difference will appear in both sum and difference phases. We can write $\cos \left(\omega_{1} t+\phi_{1}\right)+\cos \left(\omega_{2} t+\phi_{2}\right)=2 \cos (\omega t+\phi) \cos (\Delta \omega t+\Delta \phi)$ where $\omega$ and $\Delta \omega$ have the same meaning as before and $\phi=\left(\phi_{1}+\phi_{2}\right) / 2$ and $\Delta \phi=\left(\phi_{1}-\phi_{2}\right) / 2$.

General Case If the amplitudes and frequencies all differ, then the phasor diagram must be dynamic as illustrated in Fig. 2 (c), and the tip of the resultant will trace out a shape in time called a cycloid. For instance, the path followed by the tip of the resultant vector in the complex plane for different amplitudes and frequencies differing by a factor of two is plotted in Fig. 4


Figure 4: The location of the tip of the resultant vector for the two driving forces $1.5 \cos 1.2 t$ and $1.3 \cos 0.6 t$, which draws out a cycloid.

### 2.5 Normal modes of coupled oscillators

Instead of making the driving force on an oscillator a mysteriously external phenomenon (which we have implicitly done so far: we haven't allowed the oscillator to react back on the driver in the way that, say, a small child on a swing will react with their feet), what happens if it comes from another oscillator? We will tackle this problem for two oscillators in one dimension only, but the generalisation to many oscillators is rather powerful: it can be used as a model of various properties of materials, for instance. This will also introduce the important concept of normal modes, though the full power of this concept requires matrices and matrix diagonalisation which will be introduced in the second year.

Let's consider two oscillators joined by a spring of stiffness $S$ (we can imagine, for instance, two pendulums joined by a spring, or two masses connected to opposite walls of a box with springs and joined by a different spring). An example of the set-up is shown in Fig. 5


Figure 5: Example of coupled oscillators with masses $m_{1}$ and $m_{2}$. The spring joining the masses has constant $S$, while the springs joining the masses to the walls has constant $s$.

We assume that the masses are the same $\left(m_{1}=m_{2}=m\right)$ though this is not a restrictive assumption. Then we can write the equations of motion for the individual oscillators:

$$
\begin{align*}
& m \frac{\partial^{2} \psi_{1}}{\partial t^{2}}=-s \psi_{1}-S\left(\psi_{1}-\psi_{2}\right)  \tag{56}\\
& m \frac{\partial^{2} \psi_{2}}{\partial t^{2}}=-s \psi_{2}-S\left(\psi_{2}-\psi_{1}\right) \tag{57}
\end{align*}
$$

We notice two things: first, that these are just the same as equations for driven oscillators; second, that the driving terms link the equations of motion. The link makes solving the equations potentially rather more complicated that we'd find for a single oscillator.

If the coupling was not present $(S=0)$ then the two oscillators would move harmonically, with no link between them. Given that this is their fundamental behaviour it's reasonable to look for harmonic solutions to the coupled problem. For
this set of equations, we can find these harmonic solutions by noticing that adding or subtracting the equations simplifies them considerably:

$$
\begin{align*}
& m \frac{\partial^{2}}{\partial t^{2}}\left(\psi_{1}+\psi_{2}\right)=-s\left(\psi_{1}+\psi_{2}\right)  \tag{58}\\
& m \frac{\partial^{2}}{\partial t^{2}}\left(\psi_{1}-\psi_{2}\right)=-s\left(\psi_{1}-\psi_{2}\right)-2 S\left(\psi_{1}-\psi_{2}\right)=-(s+2 S)\left(\psi_{1}-\psi_{2}\right) \tag{59}
\end{align*}
$$

These are just the harmonic equations we were looking for, but we are acting on combinations of the original coordinates (which are more generally known as normal modes). We find that there are two solutions:

$$
\begin{align*}
\psi_{1}+\psi_{2} & =A_{a} \cos \left(\omega_{a} t+\phi_{a}\right)  \tag{60}\\
\psi_{1}-\psi_{2} & =A_{b} \cos \left(\omega_{b} t+\phi_{b}\right)  \tag{61}\\
\omega_{a} & =\sqrt{s / m}  \tag{62}\\
\omega_{b} & =\sqrt{(s+2 S) / m} \tag{63}
\end{align*}
$$

These can be written either in terms of $\psi_{1}$ and $\psi_{2}$ alone, or in terms of new coordinates $q_{a}$ and $q_{b}$ :

$$
\begin{align*}
\psi_{2} & =\psi_{1}=\frac{1}{2} A_{a} \cos \left(\omega_{a} t+\phi_{a}\right)  \tag{64}\\
-\psi_{2} & =\psi_{1}=\frac{1}{2} A_{b} \cos \left(\omega_{b} t+\phi_{b}\right)  \tag{65}\\
q_{a} & =\left(\psi_{1}+\psi_{2}\right) \sqrt{\frac{m}{2}}  \tag{66}\\
q_{b} & =\left(\psi_{1}-\psi_{2}\right) \sqrt{\frac{m}{2}} \tag{67}
\end{align*}
$$

The $q$ s are called mode coordinates or normal coordinates of the system. These are very useful, as they lead to uncoupled equations ( $\ddot{q} a+\omega_{a}^{2} q_{a}=0$ and equivalent for $q_{2}$ ) and also simple forms for the energies in the system:

$$
\begin{align*}
T & =\frac{1}{2} m{\dot{\psi_{1}}}^{2}+\frac{1}{2} m{\dot{\psi_{2}}}^{2}  \tag{68}\\
& =\frac{1}{2}{\dot{q_{a}}}^{2}+\frac{1}{2} \dot{q}_{b}^{2}  \tag{69}\\
V & =\frac{1}{2} s \psi_{1}^{2}+\frac{1}{2} s \psi_{2}^{2}+\frac{1}{2} S\left(\psi_{1}-\psi_{2}\right)^{2}  \tag{70}\\
& =\frac{1}{2} \omega_{a}^{2} q_{a}^{2}+\frac{1}{2} \omega_{b}^{2} q_{b}^{2} \tag{71}
\end{align*}
$$

Notice that the potential in particular contains no cross-terms when we work with the mode coordinates, whereas terms in $\psi_{1} \psi_{2}$ will appear with the original system coordinates. The use of $\sqrt{m}$ in the definition of the $q$ s helps when there are different masses, and is not necessary in this case. If we scale $\psi$ or $q$ by a constant then the solution of the equation does not change.

## 3 Basic Waves (J\&S 16)

Waves can be found in many different areas of physics, including electromagnetic fields, pressure variations in a gas, solids and stretched strings. We will consider many of these as examples of waves as we go along. However, a general definition of a wave is rather hard. Waves do not require a medium to propagate (e.g. electromagnetic waves) though many waves propagate only through a material (e.g. sound). Waves do not need to be periodic (though often are periodic). They involve the transfer of energy (though standing waves appear to violate this definition) and consist of a disturbance which propagates in time and space. We will now derive the equation for the simplest wave propagation.

### 3.1 Stretched string

Let us consider a string with mass per unit length $\mu$ under a tension $T$. We will investigate what happens when it is displaced slightly perpendicular to its length, and derive a wave equation which will turn out to be quite general. So that we know where we're going, here is the wave equation in one dimension:

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial t^{2}}=c^{2} \frac{\partial^{2} \psi}{\partial x^{2}} \tag{72}
\end{equation*}
$$



Figure 6: A stretched string displaced away from the horizontal. We have $d x=\Delta x, d m=\mu \Delta x$ and $x_{2}=x_{1}+\Delta x$.
where $c$ is the speed of the wave. We will now derive this for small displacements of our string. The set up we will consider is illustrated in Fig. 6

We assume that the vertical displacement of the string is given by $\psi$ but that the string lies initially along the $x$ axis (which will give us a transverse wave: one where the displacement is in a different direction to the propagation). We are interested in the forces acting on the string given the displacement; the force can be written, using Newton's second law, as:

$$
\begin{equation*}
F=m a=\mu \Delta x \frac{\partial^{2} \psi}{\partial t^{2}} \tag{73}
\end{equation*}
$$

What force will act on the displaced string ? It will be a component of the tension, which we will need to resolve into components along $x$ and along the displacement, as the string is no longer purely along $x$. For small displacements we could write, by inspection:

$$
\begin{equation*}
F=T\left(\frac{\partial \psi}{\partial x}\right)_{x+\Delta x}-T\left(\frac{\partial \psi}{\partial x}\right)_{x} \tag{74}
\end{equation*}
$$

where we are assuming that the tension does not vary with position (if it did, we would have a rather more complex situation involving dispersion which we will discuss briefly later in the course). Another way to arrive at this formula is to resolve the tension geometrically, which gives:

$$
\begin{equation*}
F_{\psi}=T \sin \left(\theta_{x+\Delta x}\right)-T \sin \left(\theta_{x}\right) \tag{75}
\end{equation*}
$$

But it is fairly easy to show that $\sin \theta \simeq \tan \theta$ for small values of $\theta$ (an approximation which has an error which is third order in $\theta$ - i.e. $\sin \theta-\tan \theta=\frac{1}{2} \theta^{3}$ ), and we can write $\tan \theta=\Delta \psi / \Delta x \rightarrow \partial \psi / \partial x$, which just gives us Eq. 74 again. Now we equate these two forces:

$$
\begin{equation*}
T\left(\frac{\partial \psi}{\partial x}\right)_{x+\Delta x}-T\left(\frac{\partial \psi}{\partial x}\right)_{x}=\mu \Delta x \frac{\partial^{2} \psi}{\partial t^{2}} \tag{76}
\end{equation*}
$$

If we recall the formal definition of a derivative, we can write:

$$
\begin{equation*}
\frac{\left(\frac{\partial \psi}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial \psi}{\partial x}\right)_{x}}{\Delta x}=\frac{\partial^{2} \psi}{\partial x^{2}} \text { as } \Delta x \rightarrow 0 \tag{77}
\end{equation*}
$$

If we rearrange Eq. 76p by dividing through by $\Delta x$ and $\mu$ and substituting in Eq. 777, then we find:

$$
\begin{align*}
\frac{T\left(\frac{\partial \psi}{\partial x}\right)_{x+\Delta x}-T\left(\frac{\partial \psi}{\partial x}\right)_{x}}{\Delta x} & =\mu \frac{\partial^{2} \psi}{\partial t^{2}}  \tag{78}\\
\frac{T}{\mu} \frac{\partial^{2} \psi}{\partial x^{2}} & =\frac{\partial^{2} \psi}{\partial t^{2}} \tag{79}
\end{align*}
$$

which has the form of a wave equation with velocity $\sqrt{T / \mu}$. A quick check on dimensions ( $T$ is a force with dimensions mass $(\mathrm{M}) \times$ length $(\mathrm{L}) \times$ time $^{-2}\left(\mathrm{~T}^{-2}\right)$ while $\mu$ is mass per unit length $\left(\mathrm{ML}^{-1}\right)$ so $\sqrt{T / \mu}$ has dimensions $\mathrm{LT}^{-1}$ which is a velocity) suggests that this is reasonable. Experience with elastic bands or strings suggests that the note produced gets higher with increasing tension or decreasing mass density: if the note is proportional to the velocity then our formula makes physical sense (we will see later that this is indeed what we'd expect).

What about the signs ? If the string is pulled up then the second derivative with $x$ will be negative (we'll have a maximum) which means that there is a downward force, as we'd expect (and require for sensible motion). What assumptions have we made to get to this point, and how might they break down?

- Small displacements of the string
- $T$ and $\mu$ are independent of position and displacement (this might change for large displacements)
- Resolved force requires a small angle between $x$ and $x+\Delta x$

So provided we don't make too large a displacement of the string, our derivation will hold. We will find that the same equation (though with different displacement variable, and different physical quantities contributing to the velocity) appears in many areas, such as:

- Sound (pressure waves in a gas or solid)
- Plasma (variation in electron density)
- Electromagnetic waves (fields vary as waves)
- Electrical cables (electrical charge varies with time and position)

The equation can be written as a single term operating on $\psi$ :

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial x^{2}}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \psi=0 \tag{80}
\end{equation*}
$$

where the operator in brackets is called the d'Alembertian (named after the French mathematician and physicist d'Alembert).

### 3.2 Solving the wave equation

Now that we have an equation which is obeyed by the displacement of a stretched string, we need to look for solutions. This will also allow us to understand the general behaviour of wave equations. We can start by noticing that we need the second time derivative to equal the second position derivative multiplied by $c^{2}$. This implies that we need the same functional form for $x$ and $t$ but with a multipicative constant (and dimensionally a constant with $\mathrm{LT}^{-1}$ will convert from time to distance). If we combine the time and position variables into a single variable $x-c t$ (or $x+c t$ ), we will be able to write $\psi(x, t)=f(x-c t)$ or $\psi(x, t)=g(x+c t)$. We want to check that these will satisfy the wave equation, so we write $u=x-c t$ and use the chain rule:

$$
\begin{align*}
u & =x-c t  \tag{81}\\
\psi & =f(u)  \tag{82}\\
\frac{\partial \psi}{\partial x} & =\frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x}=\frac{\partial \psi}{\partial u}  \tag{83}\\
\frac{\partial \psi}{\partial t} & =\frac{\partial \psi}{\partial u} \frac{\partial u}{\partial t}=-c \frac{\partial \psi}{\partial u}  \tag{84}\\
\frac{\partial^{2} \psi}{\partial x^{2}} & =\frac{\partial^{2} \psi}{\partial u^{2}} \frac{\partial u}{\partial x}=\frac{\partial^{2} \psi}{\partial u^{2}}  \tag{85}\\
\frac{\partial^{2} \psi}{\partial t^{2}} & =\frac{\partial^{2} \psi}{\partial u^{2}} \frac{\partial u}{\partial t}=c^{2} \frac{\partial^{2} \psi}{\partial u^{2}}  \tag{86}\\
\Rightarrow \frac{\partial^{2} \psi}{\partial t^{2}} & =c^{2} \frac{\partial^{2} \psi}{\partial x^{2}} \tag{87}
\end{align*}
$$

So this will satisfy the wave equation. The same result can be found for $\psi(x, t)=g(x+c t)$ (though in the fourth line the factor of $c$ is positive), so using the fact that the wave equation is linear, we write

$$
\begin{equation*}
\psi(x, t)=f(x-c t)+g(x+c t) \tag{8}
\end{equation*}
$$

This is, it turns out, the most general form of the solution for the wave equation. It's interesting to note that we can rewrite the wave equation itself in terms of differentials with respect to $u$ and $v$ (noting, for instance, that $\frac{\partial}{\partial x}=\frac{\partial u}{\partial x} \frac{\partial}{\partial u}+\frac{\partial v}{\partial x} \frac{\partial}{\partial v}$ ) to give:

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial u \partial v}=0 \tag{89}
\end{equation*}
$$

The most general solution for this equation is $\psi=f(u)+g(v)$ which is, or course, just Eq. 88) again. Yet another way of rewriting the wave equation is this:

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}-c \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c \frac{\partial}{\partial x}\right) \psi=0 \tag{90}
\end{equation*}
$$

which is again solved by Eq. 88 .
The second time derivative (and knowledge of the physical world) might lead us to think of a sinusoidal or periodic function as a solution, but to use this we will need a dimensionless argument. So we divide $u$ or $v$ by some characteristic length; in the case of a periodic function the distance between repeats, which we call the wavelength $\lambda$ seems most appropriate. Notice that rescaling $u$ and $v$ by a constant $(u \rightarrow C u)$ does not change the wave velocity or the wave equation. Let's go one step further, and incorporate the factor of $2 \pi$ that we know is the period of sinusoidal functionals, giving:

$$
\begin{equation*}
\psi(x, t)=f\left(\frac{2 \pi}{\lambda} x-\frac{2 \pi c}{\lambda} t\right)+g\left(\frac{2 \pi}{\lambda} x+\frac{2 \pi}{\lambda c} t\right) \tag{91}
\end{equation*}
$$

We often define two new variables:

- $k=2 \pi / \lambda$ the wavenumber (we will see later that it is actually the magnitude of a vector quantity, the wavevector)
- $\omega=2 \pi c / \lambda=2 \pi / T=c k$ the angular frequency (which is $2 \pi$ divided by the period, $T$, or $2 \pi$ multiplied by the frequency $\nu=1 / T$ )

We can now write:

$$
\begin{equation*}
\psi(x, t)=f(k x-\omega t)+g(k x+\omega t) \tag{92}
\end{equation*}
$$

where the quantity $k x \pm \omega t$ is known as the phase.
So, if we have a periodic solution with period $T$ in time and distance $\lambda$ between peaks (or troughs), when we advance the position $x$ by $\lambda$ or the time by $T$ the phase increases by $2 \pi$, which is what is required for periodic wave motion. If we look at a fixed point in space, then the time between peaks passing that point is $\lambda / c=T$. A summary of these different quantities:

- $c$ or $v$ is the velocity of the wave (the speed at which peaks or troughs move-we will encounter other velocities later, in Sec. 7
- $\lambda$ is the wavelength, the distance between peaks (or troughs) in a periodic function
- $f$ or $\nu$ is the frequency, the rate at which peaks or troughs pass a given point for a periodic function $\nu=\frac{c}{\lambda}=\frac{1}{T}$
- $\omega$ is the angular frequency $\omega=2 \pi \nu=\frac{2 \pi c}{\lambda}=k c$
- $T$ is the period which is the time taken to perform one repeat in a periodic function (i.e. from peak to peak or trough to trough) $T=\frac{1}{f}=\frac{2 \pi}{\omega}=\frac{\lambda}{c}$
- $k$ is the wave number (the magnitude of the wave vector $\mathbf{k}$ which we will encounter later) $k=\frac{2 \pi}{\lambda}=\omega / c$
- $\phi$ is an initial, constant phase


Figure 7: An illustration of amplitude $A$, wavelength $\lambda$ and period $T$ for a cosine wave $\psi(x, t)=A \cos (k x-\omega t)$. (a) $\psi$ plotted against $x$ for a fixed value of $t$; (b) $\psi$ plotted against $t$ for a fixed value of $x$.

All of these different parameters can be used in writing a wave, so long as the appropriate conversion factors are used. Motivated by our knowledge of the simple harmonic oscillator (which we know can be solved with a sinusoidal function,
or equivalently a complex exponential) and noting that a wave can be created in a stretched string by attaching a harmonic oscillator to one end we will try a solution for a periodic wave which is a sinusoidal function of $k x-\omega t$. Let's list a number of different ways that a periodic sinusoidal wave could be written:

$$
\begin{align*}
\psi(x, t)= & A \cos (k x-\omega t+\phi)  \tag{93}\\
= & \operatorname{Re}\left[A e^{i(k x-\omega t+\phi)}\right]  \tag{94}\\
= & \operatorname{Re}\left[a e^{i(k x-\omega t)}\right]  \tag{95}\\
& \left(a=A e^{i \phi}\right)  \tag{96}\\
\psi(x, t)= & A \cos \left(\frac{2 \pi}{\lambda}(x-c t)+\phi\right)=A \cos \left(2 \pi\left(\frac{x}{\lambda}-\nu t\right)+\phi\right) \tag{97}
\end{align*}
$$

Remember that, when we use the complex exponential form (which is almost always the easiest form to use) we have to take the real part when considering the physical quantity we're measuring.

### 3.3 Energy in Waves

What is the energy stored in the wave as it travels along the stretched string? We will have to think about energy per unit length. The kinetic energy (transverse to the string) is just given as usual, though we use a mass per unit length:

$$
\begin{equation*}
K E=\frac{1}{2} \mu v^{2}=\frac{1}{2} \mu\left(\frac{\partial \psi}{\partial t}\right)^{2}=\frac{1}{2} \mu A^{2} \omega^{2} \sin ^{2}(k x-\omega t) \tag{98}
\end{equation*}
$$

where we have assumed that the wave is $\psi=A \cos (k x-\omega t)$. The potential energy will come from the stretching of the string and doing work against the tension in the string. How much has a small segment $\Delta x$ been stretched ? At a point x , we can write the length of the string (using Pythagoras' theorem) as:

$$
\begin{align*}
\Delta l & =\sqrt{\Delta x^{2}+\left(\Delta x \frac{\partial \psi}{\partial x}\right)^{2}}  \tag{99}\\
& =\Delta x\left(1+\left(\frac{\partial \psi}{\partial x}\right)^{2}\right)^{\frac{1}{2}}  \tag{100}\\
& \simeq \Delta x\left(1+\frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^{2}\right) \tag{101}
\end{align*}
$$

So the change in length is approximately $\Delta l-\Delta x=\frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^{2} \Delta x$ and the work done against the tension per unit length (i.e. dividing by $\Delta x$ ) to extend it (and hence the potential energy stored) is:

$$
\begin{equation*}
U=\frac{1}{2} T\left(\frac{\partial \psi}{\partial x}\right)^{2}=\frac{1}{2} T A^{2} k^{2} \sin ^{2}(k x-\omega t) \tag{102}
\end{equation*}
$$

where we have assumed the same form for the wave as in the kinetic energy in Eq. (98). At this point, we will introduce the impedance for waves which we already saw for oscillators. It is a measure of the resistance that a wave encounters (and can be thought of as a generalisation of the idea of resistance which occurs in circuits). We will derive it properly in Sec. 4.1. but until then we will just quote the result. For the stretched string, it is defined as:

$$
\begin{equation*}
Z_{0}=\sqrt{T \mu} \tag{103}
\end{equation*}
$$

We will shortly see that it is very similar to a drag term as we've seen for harmonic oscillators. If we recall that the speed of the wave along the string is given by $c=\sqrt{T / \mu}$ then we see that $\mu=Z_{0} / c$ and $T=Z_{0} c$. So we can write the total energy per unit length (or energy density) in terms of the speed and impedance only:

$$
\begin{align*}
E(x, t) & =K E(x, t)+U(x, t)  \tag{104}\\
& =\frac{1}{2} \mu\left(\frac{\partial \psi}{\partial t}\right)^{2}+\frac{1}{2} T\left(\frac{\partial \psi}{\partial x}\right)^{2}  \tag{105}\\
& =\frac{1}{2} \frac{Z_{0}}{c}\left[\left(\frac{\partial \psi}{\partial t}\right)^{2}+c^{2}\left(\frac{\partial \psi}{\partial x}\right)^{2}\right] \tag{106}
\end{align*}
$$

This expression is actually true for any disturbance which satisfies the wave equation.
What is the rate at which energy moves along the wave (in other words the power delivered along the string) ? The force at any point $x$ due to the string to its left is approximately $-T(\partial \psi / \partial x)$, and the rate at which work is done is:

$$
\begin{equation*}
P(x, t)=-T \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t}=-Z_{0} c \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t} \tag{107}
\end{equation*}
$$

We will return to this expression in the next section when we consider travelling waves.

### 3.4 Travelling Waves

Notice that the individual solutions $\psi_{-}=f(k x-\omega t)$ and $\psi_{+}=g(k x+\omega t)$ are both good solutions for the wave equation, and we can use the fact the the equation is linear to add them together (this also gives a valid solution, which is the most general solution):

$$
\begin{equation*}
\psi=a \psi_{-}+b \psi_{+} \tag{108}
\end{equation*}
$$

This is called the principle of superposition and leads to interference effects.
Each of the solutions $\psi_{-}=f(k x-\omega t+\phi)$ and $\psi_{+}=g(k x+\omega t+\phi)$ are called travelling waves. The direction of travel is set by the relative sign of $x$ and $t$ : if the relative sign is negative (e.g. $k x-\omega t$ ) then the wave travels in the positive x direction; if it is positive (i.e. $k x+\omega t$ ) then the wave travels in the negative x direction.

We can understand this by considering a particular shape at $t=0$, which will be given by $f(k x)$. Notice that this is an arbitrary function - it doesn't have to be periodic. Then at a time $t$ later than this, if we increase $x$ by $\frac{\omega}{k} t$ we will find:

$$
\begin{equation*}
\psi\left(x+\frac{\omega}{k} t, t\right)=f\left(k\left[x+\frac{\omega}{k} t\right]-\omega t\right)=f(k x+\omega t-\omega t)=f(k x)=\psi(x, 0) \tag{109}
\end{equation*}
$$

So the shape of the wave is the same as it was at $t=0$ but translated a distance $\frac{\omega}{k} t$ along the $x$-axis. We have already written that $\omega / k=v$, so we see that the wave is translated (or travels) a distance $v t$ along the axis. This is why the solution is called a travelling wave. We can generalise away from a form with $k$ and $\omega$ by writing $f(x \pm c t)$ with the knowledge that we may need to scale the argument by an appropriate constant to get the right units. Now think about differentiating this function with respect to $t$ and $x$. We will find that:

$$
\begin{equation*}
\frac{\partial \psi}{\partial t} \mp c \frac{\partial \psi}{\partial x}=0 \tag{110}
\end{equation*}
$$

This equation is satisfied by functions of the form $\psi(x, t)=f(x \pm c t)$ which are travelling waves. Notice the close connection with the factorisation of the wave equation we gave above in Eq. 90. If either bracket in that equation is zero then the wave equation will be satisfied-a condition which can also be written in Eq. 110. Note that this is not an exact mapping: all waves satisfy a wave equation, but not all waves satisfy the travelling wave equation.

If we return to the energy density and power formulae, we see that we can eliminate one of the differentials, so that:

$$
\begin{align*}
& E(x, t)=\frac{Z_{0}}{c}\left(\frac{\partial \psi}{\partial t}\right)^{2}=c Z_{0}\left(\frac{\partial \psi}{\partial x}\right)^{2}  \tag{111}\\
& P(x, t)=-Z_{0} c^{2}\left(\frac{\partial \psi}{\partial x}\right)^{2}=-Z_{0}\left(\frac{\partial \psi}{\partial t}\right)^{2}=-c E(x, t) \tag{112}
\end{align*}
$$

So the instantaneous kinetic and potential energy densities are equal at any point on a string under tension carrying a travelling wave. The power is given by the energy density multiplied by the speed of wave propagation and travels in the direction of wave propagation.

## 4 Transverse Waves (J\&S 16, 18)

Transverse waves are waves where the displacement of the medium (e.g. the string) is perpendicular (or transverse) to the direction of wave propagation. So, for instance, the string is displaced in the $y$ direction while the wave travels in the $x$ direction.

### 4.1 Driving a wave

Until now, we have not thought about how waves start and finish: we have assumed (implicitly) that they are infinite in time and space. However, this is not how real systems work: if a wave is to continue to propagate along a string (or down an optical fibre, say) then there must be some driving force. Let's consider a string stretched to a length $L$ with ends at $x=0$ and $x=L$. We'll place a driving mechanism of some kind at $x=0$.

What must the driving mechanism do ? It needs to produce a force which balances the transverse component of the tension in the string at $x=0$. This is a driving force:

$$
\begin{equation*}
F_{D}=-T\left(\frac{\partial \psi}{\partial x}\right)_{x=0}=\frac{T}{c}\left(\frac{\partial \psi}{\partial t}\right)_{x=0}=Z_{0}\left(\frac{\partial \psi}{\partial t}\right)_{x=0} \tag{113}
\end{equation*}
$$

We have used the travelling wave equation, Eq. (110), to simplify. The force at any instant (or instantaneous force) must be proportional to the transverse velocity of the string-which closely resembles a standard drag force in a damped harmonic oscillator. Why do we have this drag ? It arises from the energy being transported by the wave: if we want to keep a constant wave motion going, we must put energy into the system. Notice that the constant of proportionality is the characteristic impedance, Eq. 103. It is a function of the system only (in this case the tension and mass density) and does not depend on the type of motion or its frequency.

### 4.2 Terminating a wave

We have just seen that creating or driving a wave along a stretched string (or, in fact, in any medium) requires energy to be put in to the system. But what about the other end of the string? What happens there ? Let's think about a finite string, and how we can make it resemble an infinite string. We know what force we need at $x=0$ to create the wave. At the end of the string, $x=L$, we need a transverse force to balance tension in the string; if we match this tension then it will be as if the end was not there. This condition can be written:

$$
\begin{equation*}
F_{L}=T\left(\frac{\partial \psi}{\partial x}\right)_{x=L}=-Z_{0}\left(\frac{\partial \psi}{\partial t}\right)_{x=L} \tag{114}
\end{equation*}
$$

So if we have some form of drag to absorb the energy being transmitted along the string, then the wave will propagate along as if the string were infinite (assuming that we have set up the driver as described above). This idea is known as impedance matching and is important in many areas, particularly electromagnetism (where we must terminate, say, a power line or an aerial correctly). Remember that this all came about because we have a finite string (or medium in general) which we want to send a wave down, with the medium behaving as if it were infinite. This means that we must put energy in at one end, and take it out at the other.

If we do not provide the right force at the end, then something different will happen. In fact, these ideas can be extended to consider boundaries between different media (e.g. tying a light string to a rope, or light going from air into water or glass). The two media will have different impedances (in the case of the string and rope, probably different mass densities even if they're under the same tension) and something interesting will happen at the boundary.

### 4.3 Reflection and Transmission (J\&S 16.4)

Imagine a string under tension, with one end tied to a solid object (like a wall) and the other end free to be driven. If we send a pulse down the string (for example by moving the free end up and then down rapidly once), it will propagate along the string as a travelling wave. What will happen when it reaches the solid wall? Intuition or experience tells us that when a wave reaches a solid object (i.e. an object with very large impedance) it tends to be reflected. A reflected wave in one dimension travels in the opposite direction to the incoming wave, so we will need both solutions for the wave equation (i.e. $\psi=\psi_{i}+\psi_{r}=f(x-c t)+g(x+c t)$, where $\psi_{i}$ is the incoming or incident wave and $\psi_{r}$ is the reflected wave). We can also deduce this need for both waves from a more general situation we'll see below.

Now, we need to work out the relationship between the two components of the wave, and we'll do this by thinking about the boundary conditions on the wave. Much of the work done in physics involves working out appropriate boundary conditions, and solutions of differential equations given appropriate boundary conditions. We will assume that the solid wall has an infinite impedance, so there can be no propagation of the wave (the velocity will be zero). The tension along the string is provided by the wall, but the tension transverse to the string at the wall must be zero (otherwise the wall would move up and down). We can therefore write that:

$$
\begin{align*}
T\left(\frac{\partial \psi_{i}}{\partial x}\right)_{x=L} & +T\left(\frac{\partial \psi_{r}}{\partial x}\right)_{x=L}=0  \tag{115}\\
\Rightarrow T\left(\frac{\partial \psi_{i}}{\partial x}\right)_{x=L} & =-T\left(\frac{\partial \psi_{r}}{\partial x}\right)_{x=L}  \tag{116}\\
\left(\frac{\partial \psi_{i}}{\partial x}\right)_{x=L} & =-\left(\frac{\partial \psi_{r}}{\partial x}\right)_{x=L} \tag{117}
\end{align*}
$$

So there must be a $180^{\circ}$ phase change on reflection; this is another way of saying that the reflected wave has the opposite sign to the incoming wave. But what happens now if, instead of a solid wall, we have another string with a different mass per unit length ? To be clear, we will assume that we have two strings of lengths $L_{1}$ and $L_{2}$ joined at
$x=0$ held under tension T. The first string has mass density $\mu_{1}$, impedance $Z_{1}=\sqrt{T \mu_{1}}$ and has a driver at its free end ( $x=-L_{1}$ ). The second string has mass density $\mu_{2}$, impedance $Z_{2}=\sqrt{T \mu_{2}}$ and has perfect termination (i.e. a drag term with impedance $Z_{2}$ ) at $x=L_{2}$.

Now consider an incident wave, starting at $x=-L_{1}$ and moving towards the joining point at $x=0$. As it propagates, there is a transverse force on the string given by:

$$
\begin{equation*}
-T\left(\frac{\partial \psi}{\partial x}\right)=Z\left(\frac{\partial \psi}{\partial t}\right) \tag{118}
\end{equation*}
$$

We assume, generally, that $\psi=\psi_{i}+\psi_{r}, x \leq 0$. Exactly at the join, the wavefunctions on the two sides must equal, so $\psi_{i}(0, t)+\psi_{r}(0, t)=\psi_{t}(0, t)$.

At the joining point, the drag exerted by the second string on the first string will be given by its impedance multiplied by the transverse velocity of the string:

$$
\begin{equation*}
F_{\mathrm{drag}}=Z_{2} \frac{\partial \psi_{t}}{\partial t}=Z_{2} \frac{\partial}{\partial t}\left(\psi_{i}+\psi_{r}\right)=Z_{2}\left(\frac{\partial \psi_{i}}{\partial t}+\frac{\partial \psi_{r}}{\partial t}\right) \tag{119}
\end{equation*}
$$

This force will be balanced by the transverse force from the string:

$$
\begin{equation*}
-T \frac{\partial \psi}{\partial x}=-T\left(\frac{\partial \psi_{i}}{\partial x}+\frac{\partial \psi_{r}}{\partial x}\right)=Z_{1}\left(\frac{\partial \psi_{i}}{\partial t}-\frac{\partial \psi_{r}}{\partial t}\right) \tag{120}
\end{equation*}
$$

Note that the change of sign between the partial derivatives comes when we use the travelling wave equation to relate the spatial and time derivatives: the incoming wave gives a minus sign (which cancels the minus sign outside the bracket) while the reflected wave gives a plus sign. We can now derive a condition relating the incident and reflected waves, by equating the two forces:

$$
\begin{align*}
Z_{1}\left(\frac{\partial \psi_{i}}{\partial t}-\frac{\partial \psi_{r}}{\partial t}\right) & =Z_{2}\left(\frac{\partial \psi_{i}}{\partial t}+\frac{\partial \psi_{r}}{\partial t}\right)  \tag{121}\\
\left(Z_{1}-Z_{2}\right)\left(\frac{\partial \psi_{i}}{\partial t}\right) & =\left(Z_{1}+Z_{2}\right)\left(\frac{\partial \psi_{r}}{\partial t}\right)  \tag{122}\\
\left(\frac{\partial \psi_{r}}{\partial t}\right) & =\frac{Z_{1}-Z_{2}}{Z_{1}+Z_{2}}\left(\frac{\partial \psi_{i}}{\partial t}\right) \tag{123}
\end{align*}
$$

If we integrate both sides with respect to time, we find that:

$$
\begin{equation*}
\psi_{r}(0, t)=\frac{Z_{1}-Z_{2}}{Z_{1}+Z_{2}} \psi_{i}(0, t) \tag{124}
\end{equation*}
$$

This gives the relation between $\psi_{i}$ and $\psi_{r}$ at the joining point of the two strings. As both waves are travelling waves, we can relate the value at one time and place to the value at another time and place:

$$
\begin{align*}
\psi_{i}(-l, t-l / c) & =\psi_{i}(0, t)  \tag{125}\\
\psi_{r}(-l, t+l / c) & =\psi_{r}(0, t)=R \psi_{i}(0, t)=R \psi_{i}(-l, t-l / c)  \tag{126}\\
R & =\frac{Z_{1}-Z_{2}}{Z_{1}+Z_{2}} \tag{127}
\end{align*}
$$

In other words, the reflected wave at $x=-l$ and $t=t+l / c$ is the same as the incident wave at the same position but at a time $2 l / c$ in the past, and scaled by $R$, which we call the reflection coefficient. Notice that the time delay is just the time it takes for the wave to travel along the string and back again.

We can also say something about the wave in the second string. We must have the following boundary condition:

$$
\begin{equation*}
\psi_{i}(0, t)+\psi_{r}(0, t)=\psi_{t}(0, t) \tag{128}
\end{equation*}
$$

where $\psi_{t}$ is the transmitted wave. This must be so, otherwise there will be a discontinuity in the string. So the transmitted wave is related to the incident wave by the following formula:

$$
\begin{align*}
\psi_{t}(0, t) & =\psi_{i}(0, t)+R \psi_{i}(0, t)  \tag{129}\\
& =(1+R) \psi_{i}(0, t)=T \psi_{i}(0, t)  \tag{130}\\
T & =1+R=\frac{2 Z_{1}}{Z_{1}+Z_{2}} \tag{131}
\end{align*}
$$

where $T$ is called the transmission coefficient. Notice that the reflection coefficient will take values between -1 and 1 , while the transmission coefficient will take values between 0 and 2 :

$$
\begin{align*}
-1 & \leq R \leq 1  \tag{132}\\
0 & \leq T \leq 2 \tag{133}
\end{align*}
$$

If there is a single string terminated with a very large impedance (a solid wall, as before, so $Z_{2} \gg Z_{1}$ ) then we will have $R=-Z_{2} / Z_{2}=-1$ and we get the phase change we saw before (and no transmission). If the string is terminated with a very small impedance (a free end, for instance, so that $Z_{1} \gg Z_{2}$ ) then we will have $R=Z_{1} / Z_{1}=1$ and no phase change in the reflected wave. When there are two strings, then in the limit of the second string having negligible impedance the transmitted wave will have twice the amplitude of the incident wave, and the reflected wave will have the same height as the incident wave. When the two impedances are matched, it will be as if there was only one string $(R=0, T=1)$.

The phase velocities and wavelengths on either side of an interface between two media will be different, but the frequencies will be the same, provided that the interface has no mechanism for driving waves. We can understand by thinking about the effect of the interface on the second medium: it will act as a harmonic driving force at frequency $\omega$. Consider a join between two strings of different mass per unit length (but under the same tension) to be specific. Then if we send a wave with frequency $\omega$ down the first string, all points on the string will oscillate harmonically with frequency $\omega$. This must be true of the junction, which will then excite waves of the same frequency in the second string. So we have:

$$
\begin{align*}
\omega_{1} & =\omega_{2}=\omega  \tag{134}\\
\nu_{1} & =\nu_{2}=\frac{\omega}{2 \pi}  \tag{135}\\
c_{1} & =\sqrt{\frac{T}{\mu_{1}}}  \tag{136}\\
k_{1} & =\frac{\omega_{1}}{c_{1}}=\frac{\omega}{c_{1}}  \tag{137}\\
\lambda_{1} & =\frac{c_{1}}{\nu}  \tag{138}\\
c_{2} & =\sqrt{\frac{T}{\mu_{2}}}  \tag{139}\\
k_{2} & =\frac{\omega_{2}}{c_{2}}=\frac{\omega}{c_{2}}  \tag{140}\\
\lambda_{2} & =\frac{c_{2}}{\nu} \tag{141}
\end{align*}
$$

Later in the course we will encounter situations where the frequency and wavelength are not so simply related (called dispersive systems) but even for these materials frequencies are conserved across boundaries.

We can summarise the results on driving waves, terminating, and behaviour at interfaces as follows:

- If we want to drive a wave along a semi-infinite string, then the driver must supply a force proportional to the transverse velocity of the end of the string it is driving. The constant of proportionality is known as the characteristic impedance.
- If we want to terminate a finite, driven string so that the same wave motion as would be found on an infinite string is supported, it must be terminated with an impedance equal to the characteristic impedance of the string.
- If a string (or any medium) is not terminated with the correct impedance then reflection as well as transmission will occur at the interface according to the reflection and transmission coefficients given above in Eq. 127) and Eq. 131 .


### 4.4 Standing Waves (J\&S 18.2, 18.3)

So far, we have considered a stretched string with a driver at one end and some form of impedance at the other end. Now we will consider a set-up where the string is fixed at both ends; we will assume that there is some means of driving a wave in the string (for instance a guitar pick or a violin bow). This is another example of boundary conditions which we can implement; again, we must have both forms of wave: $\psi=f(x-c t)+g(x+c t)$. If the string has length $L$ then we can write:

$$
\begin{align*}
\psi(0, t) & =\psi(L, t)=0  \tag{143}\\
\left(\frac{\partial \psi}{\partial t}\right)_{x=0} & =\left(\frac{\partial \psi}{\partial t}\right)_{x=L}=0 \tag{144}
\end{align*}
$$

Let us consider the general wave solution and assume that we will have a sinusoidal solution of some kind. Then we can write, quite generally,

$$
\begin{equation*}
\psi(x, t)=A e^{i(\omega t-k x)}+B e^{i(\omega t+k x)} \tag{145}
\end{equation*}
$$

Note that I've written the two so that the time variation shares the same sign and the x variation differs. This is just as general a solution of the wave equation, and makes the maths slightly easier. What can we learn from the boundary conditions? Let's look at $x=0$ first.

$$
\begin{align*}
\psi(0, t) & =A e^{i \omega t}+B e^{i \omega t}=0  \tag{146}\\
\Rightarrow A & =-B  \tag{147}\\
\psi(x, t) & =A e^{i \omega t}\left(e^{i k x}-e^{-i k x}\right)  \tag{148}\\
& =2 i A e^{i \omega t} \sin (k x) \tag{149}
\end{align*}
$$

So we've shown something quite important about the form of the wave just from the first condition; notice that the minus sign between the two components is exactly what we'd expect for a wave reflected from a solid wall. Now let's consider the boundary condition at $x=L$ :

$$
\begin{align*}
\psi(L, t) & =2 i A e^{i \omega t} \sin (k L)=0  \tag{150}\\
\Rightarrow k L & =n \pi \tag{151}
\end{align*}
$$

where $n$ is an integer, as we know that $\sin$ has zeroes for integer multiples of $\pi$. So we have a series of solutions for $k$ which will all fit the boundary conditions. We can write:

$$
\begin{align*}
k_{n} & =\frac{n \pi}{L}  \tag{152}\\
\lambda_{n} & =\frac{2 \pi}{k_{n}}=\frac{2 L}{n}  \tag{153}\\
n & =1,2,3, \ldots \tag{154}
\end{align*}
$$

So there are only a certain set of wavelengths that will fit onto the string; this makes sense, as we have imposed a certain length scale on the system, and should expect the resulting solutions to fit into that length. In fact, if we take the idea of normal modes from Sec. 2.5 to its continuous limit, we find that the allowed solutions for the string are the normal modes of the system.

The first five standing waves for a string fixed at both ends are shown in Fig. 8 The patterns along the string are important, and we can define:

- Nodes Where the wave amplitude is zero (these points are stationary at all times)
- Antinodes Where the wave amplitude is a maximum

There are $n$ maxima or anti-nodes, and $n-1$ stationary points or nodes for mode $n$. Since the speed of the wave depends on the material properties of the system (the tension and mass density in a string), the frequencies of the waves must also be arranged in a series which depends on $n$ :

$$
\begin{align*}
\omega_{n} & =c k_{n}=\frac{n \pi c}{L}  \tag{155}\\
\nu_{n} & =\frac{\omega_{n}}{2 \pi}=\frac{n c}{2 L} \tag{156}
\end{align*}
$$

These are the normal frequencies which arise because of the boundary conditions imposed on the system. Notice that, for this system (which is rather special), the normal frequencies are integer multiples of the first frequency. In general, the integer multiples of a fundamental frequency are called the harmonics.

Notice that the standing waves arise because of interference between two waves moving in opposite directions. The formation of standing waves occurs because of constructive interference between waves moving in opposite directions.


Figure 8: The first five standing waves $(n=1-5)$ for a string of length $L=5$. Waves are displayed at four different times $(\omega t=0, \omega t=\pi / 3, \omega t=2 \pi / 3, \omega t=\pi)$ to indicate the full range of displacements.

### 4.4.1 Resonance (J\&S 18.4)

A string fixed at both ends can support a series of different waves; the lowest frequency wave (with longest wavelength) is called the fundamental frequency. In the same way, most physical systems will have natural frequencies as we saw in the case of the harmonic oscillator. These frequencies are also known as resonant frequencies, as an excitation applied at this frequency will generate a resonance; as we mentioned briefly in Sec. 2.3, the amplitude of a damped, driven harmonic oscillator will be at a maximum when the driving frequency equals the natural or resonant frequency.

If a complex excitation is applied to an object, causing it to vibrate, it will tend to pick out its resonant frequencies, as these will respond with a large amplitude. A sharp impulse (like a kick or a whack from a stick) delivered to a simple oscillator will give a motion which is somewhat chaotic initially but will settle down to an oscillation at the natural frequency. The impulse consists of many frequencies (which can be investigated with Fourier analysis) but the oscillator will only respond strongly at its resonant frequency. Most objects which vibrate will have multiple resonant frequencies (depending on the boundary conditions which they impose).

### 4.4.2 Nodes and Antinodes

The wave which is supported on a string fixed at both ends have a very different form to those we used above: the time variation has been separated from the spatial variation:

$$
\begin{equation*}
\psi(x, t)=\operatorname{Re}\left[2 i A e^{i \omega t} \sin (k x)\right]=-2 A \sin (\omega t) \sin (k x) \tag{157}
\end{equation*}
$$

So each point on the string is in phase with all other points on the string, and they undergo simple harmonic motion, with the amplitude of the motion depending on the position along the string. The points on the string where $k x=n \pi$ for $n$ an integer will have zero amplitude (as $\sin (n \pi)=0$ ); these are the nodes. The points on the string where $k x=\left(n+\frac{1}{2}\right) \pi$ for $n$ an integer will have maximum amplitude of $2 A$.

The positions $x$ of the nodes and antinodes are found by substituting $k=2 \pi / \lambda$ into the expressions given above:

$$
\begin{align*}
x_{\text {node }} & =\frac{n \pi}{2 \pi / \lambda}=n \frac{\lambda}{2}  \tag{158}\\
x_{\text {antinode }} & =\frac{\left(n+\frac{1}{2}\right) \pi}{2 \pi / \lambda}=\left(n+\frac{1}{2}\right) \frac{\lambda}{2} \tag{159}
\end{align*}
$$

The nodes are separated from each other by half a wavelength (which is easy to see by calculating $x(n=1)-x(n=$ $0)$ ), and the anti-nodes are also separated from each other by half a wavelength (this is by definition: these are the zeroes and extrema of a sinusoidal function and must be separated by half a wavelength). Each node is separated by $\lambda / 4$ from the nearest anti-nodes.

As an example, consider a piano string (this is an oversimplification - pianos have two or three strings for each note). The mass per unit length is typically around $0.01 \mathrm{~kg} / \mathrm{m}$ and the tension is around 800 N (equivalent to the gravitational force exerted by the mass of an average person). The velocity of waves on this string will be $\sqrt{800 / 0.01}=283 \mathrm{~m} / \mathrm{s}$. If the wire is 0.6 m long and the ends are fixed, then the lowest allowed wavelength will be 1.2 m (with other waves with wavelengths 0.6 m and 0.4 m and so on also allowed). The frequencies of these waves will be:

$$
\begin{align*}
f_{1} & =\frac{c}{\lambda_{1}}=236 \mathrm{~Hz}  \tag{160}\\
f_{2} & =472 \mathrm{~Hz}  \tag{161}\\
f_{3} & =708 \mathrm{~Hz} \tag{162}
\end{align*}
$$

The fundamental will be just below middle C (which is about 262 Hz ).

### 4.4.3 Harmonics

The fundamental frequency for an ideal vibrating string will arise from the longest wavelength which can be supported. This is $\lambda_{1}=2 L$ for a string of length $L$. We then write:

$$
\begin{align*}
f_{1} & =\frac{c}{\lambda_{1}}=\frac{c}{2 L}  \tag{163}\\
f_{n} & =\frac{2}{\lambda_{n}}=\frac{n c}{2 L} \tag{164}
\end{align*}
$$

for $n=1,2,3, \ldots$ where $c=\sqrt{T / \mu}$. A harmonic is simply a vibration at an integer multiple of the fundamental frequency. Systems with the same boundary conditions at both ends (e.g. fixed strings, open air columns etc.) will vibrate with all harmonics of the fundamental frequency, while systems with different boundary conditions (i.e. one end fixed and one free) will vibrate with only the odd harmonics. Most musical instruments will produce some harmonics when the fundamental frequency is excited (and it is also possible to excite the harmonics only, for instance by overblowing a wind instrument or inducing a node on a stringed instrument by touching the string lightly).

### 4.4.4 Other Boundary Conditions

It is worth noting that there are other ways to set up standing waves. If a continuous wave (a sinusoidal wave, for example) is propagated along a semi-infinite string towards a fixed end, then it will reflect with a phase change of $\pi$ (or a change in sign). If we take the incoming wave to be $\psi_{i}=A \cos (k x-\omega t+\phi)$ the reflected wave will be $\psi_{r}=-A \cos (k x+\omega t+\phi)$. Then we will find:

$$
\begin{align*}
\psi(x, t) & =\psi_{i}(x, t)+\psi_{r}(x, t)  \tag{165}\\
& =A \cos (k x-\omega t)-A \cos (k x+\omega t)  \tag{166}\\
& =-2 A \sin (k x) \sin (\omega t+\phi) \tag{167}
\end{align*}
$$

which is exactly the form we saw above in Eq. 157.
This suggests that it is worth examining other boundary conditions on waves, to explore the full range of standing waves. Instead of a fixed end (or a node) we could allow the medium to have an antinode (or free end) at one or both ends. This is quite common when thinking about sound waves and we will consider pipes in detail in Sec. 5.3. For now, let us consider what an antinode at both ends would mean (think of holding a thin steel ruler lightly in the centre and wiggling it up and down: you impose a node at the centre but nothing at the ends). This is equivalent to setting $\partial \psi / \partial x=0$ (think about the transverse component of the tension - we are setting it to zero).

We will take advantage of our knowledge of the system to suggest the form:

$$
\begin{equation*}
\psi(x, t)=2 A \cos (k x) \sin (\omega t) \tag{168}
\end{equation*}
$$

We will again impose the condition that $k x=n \pi$ at $x=0$ and $x=L$ for $n$ an integer, but this time we are forcing an antinode to be at the boundaries. The allowed series of wavelengths will then be given by:

$$
\begin{align*}
k_{n} & =\frac{n \pi}{L}  \tag{169}\\
\Rightarrow \lambda_{n} & =\frac{2 \pi}{k_{n}}=\frac{2 L}{n} \tag{170}
\end{align*}
$$

which is the same set of wavelengths (really all we've done to the system is introduce a phase shift of $\pi / 2$ ). We'll consider the effect of a free end and a fixed end when we look at sound waves in pipes in Sec. 5.3

### 4.5 Going beyond one dimension

So far we have only considered waves in one dimension: the wave variable $\psi$ depends only on $x$ (and of course $t$ ). But of course there's nothing special about the $x$ direction, and we could just as easily have chosen $y$ or $z$. This suggests that we should consider three dimensions when thinking about wave motion:

$$
\begin{equation*}
\psi(x, y, z, t)=\psi(\mathbf{r}, t) \tag{171}
\end{equation*}
$$

But if the variable (e.g. displacement on a string) can depend on all three spatial dimensions, then we must also generalise the wave equation to include differentials with respect to all three dimensions. This is done by replacing $\partial^{2} / \partial x^{2}$ with $\nabla^{2}$ :

$$
\begin{equation*}
\nabla^{2} \psi(\mathbf{r}, t)=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \psi(\mathbf{r}, t) \tag{172}
\end{equation*}
$$

Of course we now have to generalise the solutions of the wave equation. Coming back to our solution where we wrote $\psi(x, t)=e^{i(k x-\omega t)}$, it's quite easy to see that this actually describes a plane (hence the term plane wave). For any given value of the phase $k x-\omega t$ all points in the $y-z$ plane passing through $x$ will share this phase. But mathematically, this plane is defined by:

$$
\begin{align*}
\mathbf{k} \cdot \mathbf{r} & =\text { constant }  \tag{173}\\
\mathbf{k} & =(k, 0,0) \tag{174}
\end{align*}
$$

with the position vector given as usual by $\mathbf{r}=(x, y, z)$. So we see that a wave which propagates along $(x, 0,0)$ can be written in terms of a wavevector $\mathbf{k}=(k, 0,0)$ with magnitude $k$. We can generalise this to write:

$$
\begin{equation*}
\psi(\mathbf{r}, t)=A e^{i(\mathbf{k} \cdot \mathbf{r}-\omega t)} \tag{175}
\end{equation*}
$$

which travels in the direction $\hat{\mathbf{k}}=\mathbf{k} /|\mathbf{k}|$ (using a unit vector to give the direction) and has wavelength $\lambda=2 \pi /|\mathbf{k}|$. The Laplacian $\nabla^{2}$ has a simple form in Cartesian coordinates:

$$
\begin{equation*}
\nabla^{2}=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial z^{2}} \tag{176}
\end{equation*}
$$

Now we can substitute Eq. (175) into Eq. (172) which gives the following condition:

$$
\begin{equation*}
k_{x}^{2}+k_{y}^{2}+k_{z}^{2}=\frac{\omega^{2}}{c^{2}} \tag{177}
\end{equation*}
$$

But as $k_{x}^{2}+k_{y}^{2}+k_{z}^{2}=|\mathbf{k}|^{2}$ we can write:

$$
\begin{equation*}
|\mathbf{k}|=\frac{\omega}{c} \tag{178}
\end{equation*}
$$

though it's quite common to write $k=|\mathbf{k}|$.
Another aspect of three dimensional waves which is important to understand is spherical waves. Point sources will emit spherical waves (that is waves which expand outwards as spheres with an amplitude which decreases as $1 / r$ or faster). When we are far from the source, these can be treated as plane waves (just as locally the surface of the earth appears to be planar). If the source is at $\mathbf{r}_{0}$ and the observer is at $\mathbf{r}$ then the direction of the wave locally is $\left(\mathbf{r}-\mathbf{r}_{0}\right) /\left|\mathbf{r}-\mathbf{r}_{0}\right|$. This approximation is valid when the following condition is fulfilled: the length scale over which we consider the wave (say $\Delta r$ ) is much smaller than the distance from the source, $r: \Delta r \ll r$. When this is true, the variation in amplitude from the $1 / r$ decrease will be negligible, and locally the wave will appear flat.

## 5 Longitudinal Waves (J\&S 17)

So far, we have only considered a wave where the disturbance is transverse (or perpendicular) to the direction of propagation of the wave. However, this is not required; in fact, waves where the disturbance is along the direction of propagation (known as longitudinal waves) are rather common. The physics of the situation is no different-we can still think of the individual points of the medium as oscillating harmonically, but instead of the oscillations lying perpendicular to the direction of wave travel, they are parallel.

Examples of longitudinal waves include sound waves (illustrated in Fig. 9, elastic waves in solids and p-waves in the earth's crust caused by earthquakes. The same wave equation governs longitudinal waves and transverse waves, though the physical processes responsible for the restoring force are different. We will start by thinking about elastic waves in a rod, and then consider sound waves.
(a)


(b)

Figure 9: An illustration of a sound wave, showing the variation of pressure schematically (top) and as a graph (bottom).

### 5.1 Elastic Waves

We will consider small distortions of a rod (or a stretched string) which involve local compressions and extensions of the rod along its axis. We will think about a short segment lying between $x_{0}$ and $x_{0}+\Delta x$ which is bounded by two planes. Each plane will undergo harmonic oscillations, though not necessarily in phase. We could, for instance, write for the location of the first plane as $x=x_{0}+\psi_{0} \cos \left(k x_{0}-\omega t\right)$. In general, we will allow a point on the rod to move from $x_{0}$ to $x_{0}+\psi$ and $x_{0}+\Delta x$ to $x_{0}+\Delta x+\psi+\Delta \psi$, with $\Delta \psi \ll \Delta x$. So as a wave passes along the rod, the short segment will be both shifted and stretched or compressed ${ }^{2}$

There are various material parameters which allow us to think about compression and expansion. Young's modulus is a measure of the stiffness of the material (we will use $Y$ though $E$ is often used). For a rod with cross-section $A$ and Young's modulus $Y$, when a force $F$ is applied along the length, the rod extends by a fraction $F /(A Y)$. We can define the local strain on the element as the change in length (or $\Delta \psi$ ) divided by the length (or $\Delta x$ ): $\epsilon=\Delta \psi / \Delta x$. The local stress is defined as the average force per unit area (which is of course equivalent to a pressure): $\sigma=F / A$; the Young's modulus is defined as the ratio of stress to strain. We can write for the element:

$$
\begin{align*}
\sigma & =Y \epsilon  \tag{179}\\
\frac{F}{A} & =Y \frac{\Delta \psi}{\Delta x} \rightarrow Y \frac{\partial \psi}{\partial x} \text { as } \Delta x \rightarrow 0  \tag{180}\\
F & =A Y \frac{\partial \psi}{\partial x} \tag{181}
\end{align*}
$$

But we will have different forces at the two ends of the element: $F$ and $F+\Delta F$ at $x$ and $x+\Delta x$. We can expand $\Delta F$ :

$$
\begin{equation*}
F+\Delta F=F+\frac{\partial F}{\partial x} \Delta x \tag{182}
\end{equation*}
$$

The excess force on the element will be $(F+\Delta F)-F$. If we assume that the rod has density $\rho$ then the mass of the element will be $\rho A \Delta x$, and we can write the net force on the element as:

$$
\begin{align*}
\rho A \Delta x \frac{\partial^{2} \psi}{\partial t^{2}} & =\Delta F=\frac{\partial F}{\partial x} \Delta x  \tag{183}\\
\rho A \frac{\partial^{2} \psi}{\partial t^{2}} & =\frac{\partial}{\partial x}\left(A Y \frac{\partial \psi}{\partial x}\right)  \tag{184}\\
\frac{\partial^{2} \psi}{\partial t^{2}} & =\frac{Y}{\rho} \frac{\partial^{2} \psi}{\partial x^{2}} \tag{185}
\end{align*}
$$

This is, of course, just the wave equation with a velocity given by $\sqrt{Y / \rho}$. Notice that the variable which undergoes wave motion is now the displacement of a small segment of the rod along its axis, which will give rise to waves of compression and expansion. We can estimate the speed of these waves for steel which has $Y=2 \times 10^{11} \mathrm{~Pa}$ and $\rho=8,000 \mathrm{~kg} \mathrm{~m}^{-3}$ giving $c=5,000 \mathrm{~ms}^{-1}$. This is a typical wave speed in a solid. But contrast this to the speed along a stretched piano wire in Section 4.4.2 which was $\sim 280 \mathrm{~ms}^{-1}$, or a factor of 20 smaller. To achieve a transverse wave speed the same as the longitudinal speed, we would need a tension divided by the cross-sectional area equal to the Young's modulus (which would pull the wire well out of the linear, elastic regime).

[^1]But this derivation is really only valid for a thin rod where we can neglect the change in cross-section. Elastic waves in a solid can consist of longitudinal waves pressure waves (as we've just described) and transverse shear waves. Elastic constants are more complex than we have considered: forces applied along different axes will lead to different expansions. If we consider an infinite bulk solid, then the freedom to expand or contract perpendicular to the applied force is removed, which increases the elastic constants and thus the wave speeds. There are a number of different elastic moduli (or elastic constants) which relate to compressions along individual axes or several axes, but this is beyond the scope of this course.

We can also look at the characteristic impedance for a rod, starting with the equation for the stress force, Eq. 180). The longitudinal waves will be travelling waves, so we can substitute $\frac{1}{c} \frac{\partial \psi}{\partial t}=\frac{\partial \psi}{\partial x}$ and write:

$$
\begin{align*}
\frac{F}{A} & =\frac{Y}{c} \frac{\partial \psi}{\partial t}  \tag{186}\\
F & =\frac{A Y}{c} \frac{\partial \psi}{\partial t}=A \sqrt{\rho Y} \frac{\partial \psi}{\partial t} \tag{187}
\end{align*}
$$

which gives us a formula for the characteristic impedance (defined as above as the force divided by the velocity):

$$
\begin{equation*}
Z_{0}=A \sqrt{\rho Y} \tag{188}
\end{equation*}
$$

Note that this now depends on the cross-sectional area of the rod. For a steel wire with area $1 \mathrm{~mm}^{2}$, using the values above, we find a characteristic impedance around $40 \mathrm{~kg} / \mathrm{s}$ for longitudinal travelling waves.

### 5.2 Waves in a Fluid (J\&S 17.1, 17.2)

Instead of Young's modulus for a fluid we must consider a different elastic modulus, the Bulk modulus; the difference arises because a fluid cannot support shear stresses. The bulk modulus measures how easily a fluid can be compressed, and is defined as the change in pressure divided by the fractional change in volume. So if a fluid at ambient pressure $P$ with volume $V$ is compressed (or expanded) to $P+d P$ with resulting volume $V+d V$, we can write:

$$
\begin{equation*}
B=-\frac{d P}{d V / V}=-V \frac{d P}{d V} \tag{189}
\end{equation*}
$$

where the minus sign ensures that increasing pressure results in decreased volume. Note that this equation is sometimes written in terms of the compressibility of the gas which is just $\kappa=1 / B$.

We can now derive the equation of motion for an element of a fluid, which will follow the derivation for longitudinal waves in a rod rather closely (see Sec. 5.1). Let's assume (for simplicity) that we have a pipe with cross-section $A$ filled with a fluid with density $\rho$ and bulk modulus $B$. As before, we will consider an element of the fluid lying between $x$ and $x+\Delta x$ which will be disturbed so that it lies between $x+\psi$ and $x+\Delta x+\psi+\Delta \psi$. We can write the change in the thickness of the element:

$$
\begin{equation*}
\Delta \psi=\frac{\partial \psi}{\partial x} \Delta x \tag{190}
\end{equation*}
$$

This corresponds to a change in volume of $A d \psi$, so that the fractional change in volume of the element is:

$$
\begin{equation*}
\frac{d V}{V}=\frac{A \Delta \psi}{A \Delta x}=\frac{\partial \psi}{\partial x} \tag{191}
\end{equation*}
$$

Now, we're looking for the force which acts on the element; this will be given by the change in pressure multiplied by the area. But we have to be a little careful about what pressure we're examining. We are interested in the change in pressure away from the ambient pressure $P$ which we will label $p$. Now we have just found the fractional change in volume (or volume strain), and can use the bulk modulus to relate this to the pressure change:

$$
\begin{align*}
B & =-V \frac{d P}{d V}  \tag{192}\\
\Rightarrow d P & =-B \frac{d V}{V}  \tag{193}\\
\Rightarrow d P & =p=-B \frac{\partial \psi}{\partial x} \tag{194}
\end{align*}
$$

where we have substituted Eq. 191) in the last line. But the difference in pressure across the element is given by:

$$
\begin{equation*}
p(x+d x)-p(x)=\left(p(x)+\frac{\partial p}{\partial x} d x\right)-p(x)=\frac{\partial p}{\partial x} d x \tag{195}
\end{equation*}
$$

We can now relate this to the acceleration experienced by the element:

$$
\begin{align*}
-\frac{\partial p}{\partial x} A d x & =\rho A d x \frac{\partial^{2} \psi}{\partial t^{2}}  \tag{196}\\
-\frac{\partial}{\partial x}\left(-B \frac{\partial \psi}{\partial x}\right) & =\rho \frac{\partial^{2} \psi}{\partial t^{2}}  \tag{197}\\
\frac{B}{\rho} \frac{\partial^{2} \psi}{\partial x^{2}} & =\frac{\partial^{2} \psi}{\partial t^{2}} \tag{198}
\end{align*}
$$

which is, again, a wave equation with velocity $c=\sqrt{B / \rho}$. This derivation has assumed nothing specific about the fluid-in particular, we have not said whether it is a liquid or a gas.

It is important to understand that, for small pressure variations, the same wave equation is obeyed by both the displacement (which we have just demonstrated) and the pressure change. From Eq. 194 we have that $p \propto \partial \psi / \partial x$. It must thus satisfy the same equation as $\psi$, so the pressure change will satisfy the wave equation as well.

The speed of sound in air at $20^{\circ} \mathrm{C}$ is about $340 \mathrm{~m} / \mathrm{s}$, though this varies with temperature and other factors such as humidity. Impedance (by analogy to elastic waves in rod) is $A B / c$ or $A \sqrt{B \rho}$. The acoustic impedance is sometimes defined as the characteristic impedance per unit area or $\sqrt{B \rho}$.

### 5.3 Standing Waves in a Fluid (J\&S 18.5)

We can have standing longitudinal waves as well as standing transverse waves (which we first saw in Sec. 4.4. Many musical instruments use these waves as the basis for generating sound. However, the boundary conditions are rather different. If we consider a pipe of air, and apply an oscillator at one end of the pipe this will generate an antinode of displacement at the end where the air is driven. If the other end is open then it will also be an antinode of displacement. Notice that the pressure, which we've shown obeys a wave equation which is the same as the equation for the displacement, is $\pi / 2$ out of phase with the displacement, so that a pipe open at both ends will have pressure nodes. In terms of displacement, the first four modes for a pipe open are shown in Fig. 10 .

The boundary condition on the open end of a pipe is that the pressure will have a node, i.e. $\partial \psi / \partial x=0$. If the pipe is closed at one end, the displacement will have a node, i.e. $\psi=0$.


Figure 10: (Left) Standing waves for the first four modes in an open pipe. (Right) Standing waves for the first four modes in a pipe with one end closed.

A pipe which is open at both ends will carry all harmonics of the fundamental frequency. The fundamental mode will have a wavelength equal to twice the length of the pipe (as was the case for a stretched string), so that we can write:

$$
\begin{align*}
\lambda_{n} & =\frac{2 L}{n}, n=1,2,3, \ldots  \tag{199}\\
f_{n} & =\frac{c}{\lambda_{n}}=\frac{n c}{2 L}, n=1,2,3, \ldots \tag{200}
\end{align*}
$$

By contrast, a pipe which is closed at one end has a displacement node enforced there, which means that the allowed standing waves are odd multiples of half-wavelengths; the first four modes for a pipe with one closed end are shown in Fig. 10. The fundamental mode will have a wavelength which is four times the length of the column, and it is not possible to fit even harmonics (which will have either nodes or antinodes at both ends) onto the boundary conditions. In this case, we can write:

$$
\begin{align*}
\lambda_{n} & =\frac{4 L}{n}, n=1,3,5, \ldots  \tag{201}\\
f_{n} & =\frac{c}{\lambda_{n}}=\frac{n c}{4 L}, n=1,3,5, \ldots \tag{202}
\end{align*}
$$

Pipe organs (as often found in churches and concert halls) label their different sets of pipes (or stops) by the timbre and the effective length of the longest pipe. The standard set of pipes are open at both ends, and are labelled as 8 ft (eight feet or about 2.4 m for the longest pipe). Given the speed of sound in air as $330 \mathrm{~m} / \mathrm{s}$, what is the frequency of the longest pipe, which corresponds to the lowest note? How long a pipe is needed to produce the same pitch if one end is closed ?

The frequency is found as $f=c / \lambda$. Now for a pipe open at both ends, the fundamental wavelength is $2 \times L=4.8 \mathrm{~m}$. This implies that the lowest note has a frequency of $330 / 4.8=69 \mathrm{~Hz}$. If the pipes were closed at one end, then the fundamental wavelength is $4 \times L$, so we would need a pipe half the length (i.e. 4 feet or about 1.2 m long). Organs which have to fit into small spaces will often use a stopped pipe (i.e. one which is closed at one end) to produce the sound of an eight foot open pipe with a four foot closed pipe.

### 5.4 Sound Measurement (J\&S 17.3)

The intensity of a wave is the power transmitted per unit area (or the rate at which energy flows through an area). This is proportional to the amplitude of the wave squared: $I \propto A^{2}$. For sound, a scale of sound level $\beta$ has been defined relative to a reference intensity, and measured in decibels (dB). We define:

$$
\begin{equation*}
\beta=10 \log _{10}\left(I / I_{0}\right) \tag{203}
\end{equation*}
$$

with $I_{0}=10^{-12} \mathrm{~W} / \mathrm{m}^{2}$, a reference intensity defined as the threshold of hearing (though of course the threshold will vary from person to person and with age). Various tables of sound levels can be found; for instance, a pneumatic drill (or jackhammer) at 1 m is 130 dB , a vacuum cleaner at 1 m is 70 dB , conversation in a room is 50 dB and a whisper at 1 m is 30 dB . The sound level for any source will decrease with distance; if the power is emitted uniformly in all directions then the intensity will decrease with at least the square of the distance.

Notice that the scale is logarithmic (and uses logarithms in base 10, not natural logarithms) and is defined relative to a reference. An increase in sound level of 1 dB is equivalent to an intensity about 1.3 times as large, while an increase of 3 dB is about twice the intensity. If we have a sound with intensity $I / I_{0}=10^{b}$ then it would be b bels louder than $I_{0}$. But bels are too coarse a unit, so it is scaled by 10 and decibels $(\mathrm{dB})$ are used. The threshold of pain is around 120 dB (or 1 $\mathrm{W} / \mathrm{m}^{2}$ ).

Two sound levels can be compared: if $I_{2}=10^{n / 10} I_{1}$ then $I_{2}$ is $n \mathrm{~dB}$ louder than $I_{1}$. Another way of writing this is to say that $I_{2}$ is $n \mathrm{~dB}$ stronger than $I_{1}$ if $n=10 \log _{10}\left(I_{2} / I_{1}\right)$. We can see this by writing:

$$
\begin{align*}
\beta_{2} & =10 \log _{10}\left(I_{2} / I_{0}\right)  \tag{204}\\
\beta_{1} & =10 \log _{10}\left(I_{1} / I_{0}\right)  \tag{205}\\
\beta_{2}-\beta_{1} & =10 \log _{10}\left(I_{2} / I_{0}\right)-10 \log _{10}\left(I_{1} / I_{0}\right)  \tag{206}\\
& =10\left[\log _{10}\left(I_{2}\right)-\log _{10}\left(I_{0}\right)-\left(\log _{10}\left(I_{1}\right)-\log _{10}\left(I_{0}\right)\right)\right]  \tag{207}\\
& =10\left[\log _{10}\left(I_{2}\right)-\log _{10}\left(I_{0}\right)-\log _{10}\left(I_{1}\right)+\log _{10}\left(I_{0}\right)\right]  \tag{208}\\
& =10\left[\log _{10}\left(I_{2}\right)-\log _{10}\left(I_{1}\right)\right]=10 \log _{10}\left(I_{2} / I_{1}\right) \tag{209}
\end{align*}
$$

For example, if I increase a sound intensity by a factor of 100 , what is the change in sound level ? We have that $I_{2} / 1_{1}=100$, so $\log _{10}\left(I_{2} / I_{1}\right)$ is 2 and the sound level change is 20 dB .

Sounds have pitch, loudness and timbre or quality. Pitch is the same as frequency-for instance, middle C is about 260 Hz . The range of audible frequencies is about $20-20,000 \mathrm{~Hz}$ though this varies with person and age. Loudness is a quality perceived by ear, and is not fully understood. A rough rule of thumb is that loudness doubles with increase in intensity of a factor of 10 . Timbre is related to the harmonics excited, the shape of a pulse (the attack and decay and any vibrato or variation).

## 6 Doppler Effect (J\&S 17.4)

Waves will have both sources and observers (or, if you prefer, detectors). If the source or the observer are moving, then the frequency perceived by the observer will change relative to the frequency of the source. (It is important to realise that the definition of a moving source or moving observer is only unambiguous if the wave propagates through a medium; in this case, the motion we are concerned about is relative to the medium, while for electromagnetic waves in a vacuum the relative motion of the source and observer is all that matters.) This change in the frequency is known as the Doppler

Effect or Doppler Shift, and it applies to all types of waves. The standard (or Newtonian) Doppler effect requires us to consider three different cases: the source moving; the observer moving; and both moving. The waves will be emitted with frequency $f$, wavelength $\lambda$ and speed $v=\lambda f$; this is illustrated in Fig. 11 a) for a stationary observer.


Figure 11: (a) Waves emitted by a stationary source at a fixed frequency. (b) Waves emitted by a moving source at the same frequency.

### 6.1 Moving Observer

If the observer moves with speed $v_{O}$, then the apparent speed of the waves will change to $v^{\prime}=v+v_{O}$ (we assume that the sign of $v_{O}$ is such that if the observer moves towards the source it is positive). Thus the observed frequency will change:

$$
\begin{align*}
f^{\prime} & =\frac{v^{\prime}}{\lambda}=\frac{v+v_{O}}{\lambda}  \tag{210}\\
& =\frac{v+v_{O}}{\lambda} \frac{v}{v}=\frac{v+v_{O}}{v} \frac{v}{\lambda}  \tag{211}\\
& =\left(\frac{v+v_{O}}{v}\right) f=\left(1+\frac{v_{O}}{v}\right) f \tag{212}
\end{align*}
$$

Physically, an observer moving towards the source will encounter the wave peaks (or troughs) more frequently than a stationary observer. Similarly an observer moving away from the source will encounter the wave peaks less frequently; these changes in frequency are reflected in the formula just given.

### 6.2 Moving Source

If the source moves with speed $v_{S}$ then the spacing between waves will change (decreasing in the direction the source moves and increasing in the opposite direction) as illustrated in Fig. 11(b). This will lead to a change in the measured wavelength, and as the velocity of the waves is unchanged, the frequency will change. In one period, the source will move $v_{S} T$ which is also the change in wavelength, $\Delta \lambda=v_{S} T$; if the source moves towards the observer the wavelength decreases. Then we can find the changed frequency:

$$
\begin{align*}
\lambda^{\prime} & =\lambda-\Delta \lambda=\lambda-v_{S} T  \tag{213}\\
& =\lambda-\frac{v_{S}}{f}=\frac{v}{f}-\frac{v_{S}}{f}  \tag{214}\\
f^{\prime} & =\frac{v}{\lambda^{\prime}}=\frac{v}{\frac{1}{f}\left(v-v_{S}\right)}  \tag{215}\\
& =f\left(\frac{v}{v-v_{S}}\right)=f\left(\frac{1}{1-v_{S} / v}\right) \tag{216}
\end{align*}
$$



Figure 12: Source moving faster than the wave speed in a medium

### 6.3 Moving Observer \& Source

When both observer and source are moving, then both the effects described above are seen: the observed wavelength changes as does the velocity of the waves relative to the observer. Using the ideas developed above, we can write:

$$
\begin{align*}
f^{\prime} & =\frac{v^{\prime}}{\lambda^{\prime}}=\frac{v+v_{O}}{\frac{1}{f}\left(v-v_{S}\right)}  \tag{217}\\
& =\left(\frac{v+v_{O}}{v-v_{S}}\right) f \tag{218}
\end{align*}
$$

### 6.4 More Advanced Ideas

The basic treatment which we have given is correct, but there are some refinements that we can introduce.

### 6.4.1 Relative Velocities

The derivations given above assumed that the source was travelling directly towards the observer (or vice versa). In this case, the observed frequency will be raised to a constant level when the source is approaching and dropped to a constant level when it is receding. Experience of the Doppler effect (e.g. a siren on an ambulance coming towards us) suggests that when the motion is not directly along the line joining the source and the observer, there is a changing frequency.

The frequency changes because the Doppler shift only takes into account the component of the velocity of the source and/or observer along the line joining the two. Let's consider a moving source for definiteness. If the source has velocity $\mathbf{v}_{S}$ and position $\mathbf{r}_{S}$, and the observer has position $\mathbf{r}_{O}$ then we must replace $v_{s}$ in the formulae above with:

$$
\begin{equation*}
\frac{\mathbf{v}_{S} \cdot\left(\mathbf{r}_{S}-\mathbf{r}_{O}\right)}{\left|\mathbf{r}_{S}-\mathbf{r}_{O}\right|}=v_{S} \cos \theta_{S O}(t) \tag{219}
\end{equation*}
$$

where $\theta_{S O}(t)$ is the angle between the velocity and the line between the observer and source, and depends on time. The same considerations apply to moving observers.

### 6.4.2 Relativistic Doppler Effect

The Doppler effect can be derived within the framework of special relativity (rather than Newtonian Mechanics as we have just done). This will be covered in detail in PHAS1246; for now, I will just quote the result (which comes from considering the time between wavefronts when the source and observer are moving relative to each other, and transforming from the source's reference frame to the observer's).

$$
\begin{equation*}
f_{\mathrm{obs}}=\sqrt{\frac{1-v / c}{1+v / c}} f_{\mathrm{source}} \tag{220}
\end{equation*}
$$

where $v$ is the relative speed of the source and observer and $c$ is the speed of light; the signs given here are in line with those used in PHAS1246 and are for a source receding from the observer. Note that $\beta=v / c$ is often used here instead.

### 6.4.3 Shock Waves

If the source is moving faster than the velocity of the waves in the medium, then a new phenomenon emerges: a shock wave. This is illustrated in Fig. 12

The angle formed by the wavefronts as they add to each other leads to a shockwave. This is constructive interference, and can lead to large pressure variations in a gas.

## 7 Dispersive Waves

So far, we have considered waves where there is a linear relationship between angular frequency and wavenumber ( $\omega=$ $k c$ ), which are known as non-dispersive waves. When this relationship changes there will be different velocities for different frequencies, which is known as dispersion, and waves which have this property are called dispersive waves.

### 7.1 Superposition and Wave Packets (J\&S 40.6)

Before we come to dispersion, however, we must consider superposition and wave packets. We looked at the phenomenon of beats for a harmonic oscillator in Sec. 2.4 , and saw that the sum of two oscillations gave an oscillation at the average frequency modulated by an envelope at the difference frequency. We can also combine waves: this is known as superposition. When the peaks and/or troughs of the waves coincide, we have constructive interference between waves (and looking back at Fig. 3 (a), it is clear that there is constructive interference around $x=0$ and $x=31$ ). When the peaks of one wave coincide with the troughs of another, we have destructive interference (which can be seen in Fig. 3(a) around $x=16$ and $x=47$ ). As well as the simple periodic phenomenon of beats, we can form wave packets of arbitrary shape by adding up waves of several (or many) different frequencies. In general, an arbitrarily shaped packet or wave can be represented as the sum of sinusoidal waves of different frequency multiplied by frequency-dependent coefficients. An example is the square wave, which can be represented by the following sum:

$$
\begin{align*}
f(x) & =\frac{1}{2}+\sum_{n=1,2,3, \ldots} 2 c_{n} \cos (n x)  \tag{221}\\
c_{n} & = \begin{cases}(-1)^{(n-1) / 2} \frac{1}{n \pi} & n \text { odd } \\
0 & n \text { even }\end{cases} \tag{222}
\end{align*}
$$



Figure 13: Graphs showing how a square wave is built up by a sum of cosines. From left to right and top to bottom we have two, five, ten, twenty, forty and one hundred non-zero terms in the series in Eq. 221 .

This is illustrated in Fig. 13 It is also possible to represent a finite wavepacket but this requires an integral over frequency rather than a sum. The full treatment of these ideas is known as Fourier theory or Fourier analysis, and is dealt with in PHAS2246. For a wave to carry a signal (for instance speech or radio signals), we need at least two frequencieswith just one frequency, the wave is always there, while with two frequencies, the amplitude varies (at a simple level at the beat frequency).

We can define the sum of two waves, $\psi_{1}(x, t)=A \cos \left(k_{1} x-\omega_{1} t\right)$ which has phase velocity $v_{p 1}=\omega_{1} / k_{1}$ and $\psi_{2}(x, t)=A \cos \left(k_{2} x-\omega_{2} t\right)$ which has phase velocity $v_{p 2}=\omega_{2} / k_{2}$.

$$
\begin{align*}
\psi(x, t) & =A \cos \left(k_{1} x-\omega_{1} t\right)+A \cos \left(k_{2} x-\omega_{2} t\right)  \tag{223}\\
& =2 A \cos \left[\frac{\left(\omega_{1}+\omega_{2}\right)}{2} t-\frac{\left(k_{1}+k_{2}\right)}{2} x\right] \cos \left[\frac{\left(\omega_{1}-\omega_{2}\right)}{2} t-\frac{\left(k_{1}-k_{2}\right)}{2} x\right] \tag{224}
\end{align*}
$$

In the combined wave, we have the average frequency $\left(\omega_{1}+\omega_{2}\right) / 2$, which is known as the carrier frequency, and the difference frequency $\left(\omega_{1}-\omega_{2}\right) / 2$, which is known as the envelope frequency. An important question to examine is whether the two carrier and envelope waves move with the same velocity. If we assume that $\omega_{1}$ and $\omega_{2}$ are nearly the same ( or $\omega_{1}=\omega_{2}+\delta \omega$ ), then $k_{1}$ and $k_{2}$ will be nearly the same (and $k_{1}=k_{2}+\delta k$ ) and the velocity of the average will be $v_{p} \simeq v_{p 1} \simeq v_{p 2}=\omega / k$. The velocity of the difference will be given by:

$$
\begin{equation*}
\frac{\omega_{1}-\omega_{2}}{k_{1}-k_{2}}=\frac{\delta \omega}{\delta k} \tag{225}
\end{equation*}
$$

This suggests that the velocity of the envelope might related to the differential of $\omega$ with respect to $k$ (certainly as we take the difference between the two waves to zero). We will prove this below, but for now we define the group velocity as:

$$
\begin{equation*}
v_{g}=\frac{d \omega}{d k} \tag{226}
\end{equation*}
$$

This is the velocity of the envelope (or points of constant amplitude)

### 7.2 Dispersion

As mentioned above, we have so far considered situations where the phase velocity is constant, so that there is a linear relationship between $\omega$ and $k$. However, it is clear from everyday life that this is not always true: a rainbow (or the diffraction of white light by a prism) shows that different frequencies or colours travel with different velocities (otherwise they would not spread out). We defined the phase velocity as the velocity of points of constant phase, $v_{p}=\omega / k$ (remembering that the phase of a wave is given by $k x-\omega t+\phi$ ). We must also define the velocity of the envelope (or points of constant amplitude) which is known as the group velocity $v_{g}=d \omega / d k$. There will be occasions when the relationship between $\omega$ and $k$ is not linear, and the phase and group velocities will differ. In these situations, the peaks and troughs of the carrier frequency (the sum for two frequencies) will move at a different velocity to the maxima and zeroes of the envelope frequency. The shape of a wavepacket or any superposition of waves will change (typically narrow distributions will spread). This is known as dispersion, and is much more common than non-dispersive behaviour.

We can write the relationship between $v_{p}$ and $v_{g}$ as:

$$
\begin{align*}
v_{p} & =\frac{\omega}{k} \Rightarrow \omega=k v_{p}  \tag{227}\\
v_{g} & =\frac{d \omega}{d k}=\frac{d}{d k}\left(k v_{p}\right)  \tag{228}\\
& =v_{p}+k \frac{d v_{p}}{d k} \tag{229}
\end{align*}
$$

So, if the relationship connecting angular frequency and wavenumber is non-linear, the phase and group velocities will differ. Typically this will lead to attenuation (or weakening) of the wave as energy is dissipated.

### 7.3 Dispersion Relation

The algebraic relation between angular frequency, $\omega$, and wavenumber, $k$, is known as the dispersion relation. So, nondispersive waves have $v_{p}=v_{g}$ and a dispersion relation written:

$$
\begin{equation*}
k=\omega / c \tag{230}
\end{equation*}
$$

If the phase velocity varies with wavelength (or wavenumber or frequency) the wave system is said to be dispersive, and the phase velocity and group velocity will differ. Why might this happen ? If we think about a stretched string, then there might be friction at the end points, though it is easier to imagine applying damping by immersing the string in a viscous fluid. We know from the damped harmonic oscillator that we can write such a damping term as $-b \partial \psi / \partial t$. This will change the wave equation, adding to the restoring force from the tension:

$$
\begin{equation*}
\mu \frac{\partial^{2} \psi}{\partial t^{2}}=T \frac{\partial^{2} \psi}{\partial x^{2}}-\beta \frac{\partial \psi}{\partial t} \tag{231}
\end{equation*}
$$

where we have defined $\beta$ as a damping per unit length. Will a sinusoidal wave still be a solution of this new equation? If we substitute $\psi(x, t)=A e^{i(k x-\omega t)}$ into the new wave equation, we can find out. We will use the results $\partial^{2} \psi / \partial x^{2}=-k^{2}$,
$\partial^{2} \psi / \partial t^{2}=-\omega^{2} \psi$ and $\partial \psi / \partial t=-i \omega \psi$. These come from differentiating the complex exponential, and you should how to derive them.

$$
\begin{align*}
-\omega^{2} \mu \psi & =-T k^{2} \psi+i \omega \beta \psi  \tag{232}\\
\Rightarrow \omega^{2} & =\frac{T}{\mu} k^{2}+i \frac{\beta}{\mu} \omega  \tag{233}\\
c^{2} k^{2} & =\omega^{2}+i \Gamma \omega  \tag{234}\\
k & = \pm \frac{1}{c} \sqrt{\omega^{2}+i \Gamma \omega}  \tag{235}\\
k & = \pm \frac{\omega}{c} \sqrt{1+i \frac{\Gamma}{\omega}} \tag{236}
\end{align*}
$$

where we have defined $\Gamma=\beta / \mu$. So we find two things: first, we can find the dispersion relation by substituting a sinusoidal wave into the wave equation; and second, for this case, the dispersion relation will be non-linear. The nonlinearity will depend on the term $\Gamma / \omega$. If $\Gamma / \omega \ll 1$, then we can approximate $\sqrt{1+i \Gamma / \omega}=1+i \Gamma /(2 \omega)$. The wavenumber becomes complex: we have $k= \pm(a+i b)$ and $a=\omega / c$ and $b=\Gamma / 2 c$. This introduces an attenuation to the position dependence of the wave (note that we can choose $k$ to be positive or negative, and we will choose the sign so that the attenuation term is an attenuation, not an amplification which is unphysical). Notice the similarity to the damped harmonic oscillator, where the drag term leads to temporal attenuation. We can write:

$$
\begin{equation*}
\psi(x, t)=A e^{\Gamma x / 2 c} e^{i\left(\frac{\omega}{c} x-\omega t\right)} \tag{237}
\end{equation*}
$$

So the motion of the string will be the expected sinusoidal response at the driving frequency but with a spatial attenuation which depends on the damping applied and the speed of the wave along the string. There are many other examples of dispersive systems: stiff or lumpy (non-uniform) strings, water waves, and electromagnetic waves in dielectric media for example. We will consider water waves in more detail in Sec. 7.4

We will now offer a more formal demonstration for why the group velocity is written as $d \omega / d k$. The phase velocity is found for points which obey the condition:

$$
\begin{equation*}
k x-\omega t+\phi=\text { constant } \tag{238}
\end{equation*}
$$

In other words, points for which the phase is constant. If we differentiate this with respect to time, we quickly find that $d x / d t=v_{p}=\omega / k$; note that we are solving for points on the wave which obey the condition in Eq. 238; those points move at the phase velocity.

What about the group velocity? This can be considered as the velocity of points of constant amplitude: we want to follow the envelope not the waves inside the envelope. To be clear: we are considering a wave packet of some kind formed by the superposition of different waves of different frequency (and with different amplitudes). Let's consider an envelope which has a well-defined maximum which we can follow; that maximum will be the point at which all components are constructively superposed (i.e. they will all add). To be constructively superposed, then all components must have the same phase $k x-\omega t+\phi$, though the individual parts of the phase (in particular $k x$ and $\omega t$ ) will be different. To find the group velocity, we seek the set of points where $k x-\omega t+\phi$ is the same for all frequencies-that is, independent of frequency. We can write this condition as:

$$
\begin{align*}
\frac{d}{d \omega}[k x-\omega t+\phi] & =0  \tag{239}\\
\frac{d k}{d \omega} x-t & =0  \tag{240}\\
\Rightarrow v_{g} & =\frac{d \omega}{d k} \tag{241}
\end{align*}
$$

When the group velocity is less than the phase velocity, this is called normal dispersion. When it is greater than the phase velocity, it is called anomalous dispersion. These terms come from optics (where normal dispersion is most common, and where transmission of energy or information at speeds greater than that of light in a vacuum is not possible).

This brief look at dispersion has only scratched the surface of a rich, complex subject. In the next section we will examine water waves as an example of dispersive waves. They show a complex behaviour with different dispersion relations for different wavelengths, and behaviour which also depends on the depth of the water.

### 7.4 Water Waves

Water waves are an important area of study (both for the interesting physics they show, but also their importance in design of structures on the coast and implications for events such as tsunami). They are surface waves, and combine transverse and longitudinal motion, and have a complicated dispersion relation which depends on the relation between the water
depth and the wavelength. We will not derive the formulae given here; you can find an excellent discussion in Main, Chapter 13, for instance.

We make two key assumptions about the water:

- It is hard to compress
- It has low viscosity

The second assumption can be relaxed, and will lead to attenuation. The first assumption explains how the water molecules move in a water wave and is important. Consider the trough of a water wave, where there is a downward displacement of the water; as the water is hard to compress, some of the water must have flowed sideways to a region where the surface displacement is upwards (or a peak). This tells us that the wave will contain transverse and longitudinal components (and arises because it is a wave at the surface of the water). So we will have to write the wave motion as both in $x$ (along the motion) and $y$ (transverse to the motion). Since we have a boundary (the surface) it is also reasonable to assume that the displacement will depend on distance from the surface, so that we will have $\psi_{x}(x, y, t)$ and $\psi_{y}(x, y, t)$. We will also assume a sinusoidal wave (from observations of bodies of water, and our experience with waves so far) and only consider the steady state.

Since the motion is sinusoidal, we can assume that both $x$ and $y$ motion will be sinusoidal, and ask what the phase difference will be. Picture the point on a wave between the trough and the crest (i.e. where the $y$ displacement is zero). The slope here is greatest, which suggests that the longitudinal displacement will be at a maximum (most water rearranged). This simple argument suggests that the phase difference between the $x$ and $y$ components of the displacement is $\pi / 2$; this implies that the motion of the particles in the wave will be elliptical or circular. Observations of something floating on the sea (whether a swimmer or a seagull) as waves pass suggest that a roughly circular motion is correct. We will give detailed forms for the displacements after considering the dispersion relation (which we will quote without proof).

Dispersion Relation The dispersion relation for water waves of any depth can be shown to be:

$$
\begin{align*}
\omega^{2} & =\left(g k+\frac{\gamma k^{3}}{\rho}\right) \tanh (k h)  \tag{242}\\
\omega^{2} & =k^{2}\left(\frac{g}{k}+\frac{\gamma k}{\rho}\right) \tanh (k h) \tag{243}
\end{align*}
$$

where $g$ is acceleration due to gravity, $\gamma$ is the surface tension and $\rho$ is the density.
There are two sources of restoring force acting on the water waves: one which depends on the surface tension, which tends to flatten out curves; and one which depends on gravity, which represents the tendency of the water piled up in the crests to fall under gravity. The two terms depend differently on $k$, and have equal magnitude when:

$$
\begin{equation*}
\frac{g}{k}=\frac{\gamma k}{\rho} \Rightarrow k^{2}=\frac{g \rho}{\gamma} \tag{244}
\end{equation*}
$$

or at a wavelength $\lambda=2 \pi \sqrt{\gamma /(\rho g)}$. Waves with a smaller wavelength than this will be governed by surface tension effects, while waves with a larger wavelength will be governed the gravity term. For water, $\gamma=0.073 \mathrm{~N} / \mathrm{m}, \rho=1,000 \mathrm{~kg} / \mathrm{m}^{3}$ which, with $g=9.8 \mathrm{~m} / \mathrm{s}^{2}$, gives $\lambda=0.017 \mathrm{~m}$ (or 17 mm ). Waves shorter than this (known as ripples) will be dominated by surface tension.

For gravity waves, there are two regimes which are important, which depend on the relative magnitudes of the wavelength, $\lambda$, and the water depth, $h$. Shallow water is defined as $k h \ll 1$ (with $k=2 \pi / \lambda$ as usual) while deep water is defined as $k h \gg 1$. We will give formulae for the displacement of particles in the water for these two extremes without proof, and discuss the dispersion relation in these two cases.

Deep Water In deep water $(k h \gg 1$ and $|y| \ll h)$, we can write:

$$
\begin{align*}
\psi_{x}(x, y, t) & =A e^{k y} \sin (\omega t-k x)  \tag{245}\\
\psi_{y}(x, y, t) & =A e^{k y} \cos (\omega t-k x) \tag{246}
\end{align*}
$$

This motion is illustrated in Fig. 14 We see that particles describe circles, with the radius depending on depth. Once $h>\lambda$ there is little motion, which explains how structures such as oil rigs achieve stability: if the buoyant part is sufficiently deep, it will not be affected by the surface motion.

We can examine the dispersion relation for this limit. If $k h \gg 1$, $\tanh \rightarrow 1$, so we can write:

$$
\begin{equation*}
\omega^{2}=k^{2}\left(\frac{g}{k}+\frac{\gamma k}{\rho}\right) \tag{247}
\end{equation*}
$$

We then have to consider the two terms.


Figure 14: The displacement of particles in a water wave; A indicates waves in deep water and B waves in shallow water.

For ripples ( $\lambda$ small) the dispersion relation becomes:

$$
\begin{equation*}
\omega=\sqrt{\frac{\gamma}{\rho}} k^{\frac{3}{2}} \tag{248}
\end{equation*}
$$

This gives us $v_{p}=\sqrt{k \gamma / \rho}$. Differentiating, we get $v_{g}=\frac{3}{2} \sqrt{k \gamma / \rho}=\frac{3}{2} v_{p}$, so the group velocity is larger than the phase velocity.

For long wavelengths, we have gravity waves. The dispersion relation becomes:

$$
\begin{equation*}
\omega=\sqrt{g k} \tag{249}
\end{equation*}
$$

The phase velocity is then $v_{p}=\sqrt{g / k}$ and the group velocity $v_{g}=\frac{1}{2} \sqrt{g / k}=\frac{1}{2} v_{p}$, which is smaller than the phase velocity. Notice that the dispersion relation does not contain the density, so this relation will hold for any non-viscous, incompressible fluid. A good example is the ocean swell, which has wavelengths typically up to 100 m . This gives the phase velocity around $10 \mathrm{~m} / \mathrm{s}$, though the energy will travel at the group velocity.

Shallow Water In shallow water ( $k h \ll 1$ ), we can write:

$$
\begin{align*}
\psi_{x}(x, y, t) & =\frac{A}{k h} \sin (\omega t-k x)  \tag{250}\\
\psi_{y}(x, y, t) & =A(1+y / h) \cos (\omega t-k x) \tag{251}
\end{align*}
$$

This motion is illustrated in Fig. 14. We see that particles describe ellipses, with the horizontal (longitudinal) amplitude almost the same at all depths and the vertical amplitude falling off fast with depth, so that these waves are close to longitudinal (note that $y<0$ and, by definition, $0>y>-h$ ). Since we have $k h \ll 1$, we can expand out tanh $k h \simeq$ $k h-(k h)^{3} / 3$, and discard the term in $k^{3}$ in Eq. 242, so that the dispersion relation is:

$$
\begin{equation*}
\omega^{2}=g h k^{2}\left(1-\frac{h^{2} k^{2}}{3}\right) \tag{252}
\end{equation*}
$$

For small values of $h k$ we can simplify even further, and write $\omega \simeq \sqrt{g h} k$ so that $v_{p}=v_{g}=\sqrt{g h}$. Thus the speed of the waves decreases as the water gets shallower. These waves are non-dispersive, and as the power must be constant the decrease in speed will result in an increase in the amplitude. For the waves seen at the sea-shore, this eventually results in the waves breaking. A more dramatic and potentially more destructive example is that of a tsunami, where a massive but slow upheaval of the sea bed generates a wave with a very long wavelength. For a sea depth of 4 km , the vvelocity will be around $200 \mathrm{~m} / \mathrm{s}$, though the amplitude will often be less than 1 m . As the depth decreases, so does the velocity and the amplitude of the wave increases; as there is little dispersion, the result is a few extremely powerful wave crests with destructive and tragic results.

## 8 Summary

In this section, I provide a brief overview of the most important ideas from the waves and acoustics part of the course, given as bullet points. This list is not definitive and you should be aware of all the material covered in the lectures.

- The general equation for a harmonic oscillator (Sec. 22 is:

$$
\begin{equation*}
m \frac{d^{2} \psi}{d t^{2}}=-s \psi-b \frac{d \psi}{d t}+F_{0} \cos \omega_{f} t \tag{253}
\end{equation*}
$$

where $m$ is the mass of the oscillator, $s$ is a stiffness (and gives the restoring force), $b$ is a resistance or damping and the driving force $F_{0}$ oscillates at frequency $\omega_{f}$

- Phasors are used to represent complex exponentials, $\psi(t)=A e^{i(\omega t+\phi)}$, which describe the motion of a harmonic oscillator
- A simple harmonic oscillator will respond at a frequency $\omega=\omega_{0}=\sqrt{s / m}$
- A driven harmonic oscillator will respond at the driving frequency $\omega_{f}$ in the steady state
- The impedance is defined as the amplitude of the driving force divided by the complex amplitude of the oscillator velocity, $F_{0} / \dot{\psi}$
- Two (or more) oscillations can be added to give a resulting oscillation
- With the same frequency, the resultant can be found using a phasor diagram or complex exponential arithmetic
- With different frequencies, the phenomenon of beats is found:

$$
\begin{equation*}
\psi(t)=A \cos \omega_{1} t+A \cos \omega_{2} t=2 A \cos \omega t \cos \Delta \omega t \tag{254}
\end{equation*}
$$

where $\omega=\left(\omega_{1}+\omega_{2}\right) / 2$ and $\Delta \omega=\left(\omega_{1}-\omega_{2}\right) / 2$

- Normal modes (Sec. 2.5) are collective, harmonic motions of coupled oscillators
- When we couple two (or more) oscillators with another simple harmonic force their equations of motion become linked
- By considering combinations of the oscillators (for two, the sum and difference motions) we find simple harmonic solutions
- The wave equation (Sec. 3) is:

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial t^{2}}=c^{2} \frac{\partial \psi^{2}}{\partial x^{2}} \tag{255}
\end{equation*}
$$

where $c$ is the speed of points of constant phase, or phase velocity

- We can derive this for a stretched string and find that $c=\sqrt{T / m u}$ with $T$ the tension and $\mu$ the mass per unit length
- The most general solution for the wave equation is $\psi(x, t)=f(x-c t)+g(x+c t)$
- When there is periodic motion, we write $\psi(x, t)=f(k x-\omega t)+g(k x+\omega t)$ with $k=2 \pi / \lambda$ and $\lambda$ the wavelength (or distance between two peaks or two troughs) and $\omega=2 \pi f=2 \pi / T$ is the angular frequency, $f$ is the frequency and $T$ the period (or the time interval between two peaks or troughs)
- We have $\psi(x, t)=A e^{i(k x-\omega t+\phi)}$ for a general periodic wave
- There is energy associated with a wave: for a stretched string, the potential energy is $\frac{1}{2} T A^{2} k^{2} \sin ^{2}(k x-\omega t)$ and the kinetic energy is $\frac{1}{2} \mu A^{2} \omega^{2} \sin ^{2}(k x-\omega t)$
- One of the two solutions $\psi=f(x-c t)$ or $\psi=g(x+c t)$ is called a travelling wave (Sec. 3.4), with the direction of travel given by the sign between $x$ and $t$.
- The impedance of a stretched string, $Z_{0}=\sqrt{T \mu}$
- To create a wave, a driving force $F_{D}=Z_{0}(\partial \psi / \partial t)$ must be applied
- To terminate a wave, a damping force or load $F_{L}=Z_{0}(\partial \psi / \partial t)$ must be applied
- At a boundary between different impedances we can get reflection and transmission, with $R=\left(Z_{1}-Z_{2}\right) /\left(Z_{1}+Z_{2}\right)$ the reflection coefficient and $T=1+R$ the transmission coefficient (Sec.4.3)
- Standing waves (Sec.4.4) arise when a wave is confined to a finite area with free or fixed boundary conditions
- For a stretched string of length L with fixed ends, we have $\psi(x, t)=2 A \sin \omega t \sin k_{n} x$, with $k_{n}=n \pi / L$ for $n=1,2,3, \ldots$
- Every point on the string moves in phase; the points with zero displacement are nodes and the points with maximum displacement are antinodes
- Free ends give the same wavenumbers as fixed ends; however, if one end is fixed and the other free, we have $k_{n}=n \pi / 2 L$ for $n=1,3,5, \ldots$
- Longitudinal waves (Sec. 5 ) have the displacement $\psi$ in the direction of the wave travel
- On an elastic rod, the wave motion consists of compression and expansion of the rod
- The same wave equation is obeyed, but with different speeds.
- For elastic waves on a rod with cross-sectional area $A$, density $\rho$ and Young's modulus $Y, c=\sqrt{Y / \rho}$ and $Z_{0}=$ $A \sqrt{\rho Y}$
- In a fluid with bulk modulus $B$ and density $\rho, c=\sqrt{B / \rho}$
- We can have standing waves in a fluid (e.g. a gas in a pipe) exactly as described above
- For an intensity (power/area) of $I_{1}$, the sound level in dB is defined as $\beta=10 \log _{10}\left(I_{1} / I_{0}\right)$, with $I_{0}=10^{-12} \mathrm{~W} / \mathrm{m}^{2}$
- If a sound level $\beta_{2}$ is $n \mathrm{~dB}$ greater than $\beta_{1}$, then $I_{2}=10^{n / 10} I_{1}$
- A moving source and a moving observer will both lead to a change in the frequency observed (Sec. 6)
- Moving observer: $f^{\prime}=\left(1+v_{O} / v\right) f$ for a wave moving with velocity $v$ and an observer moving with velocity $v_{O}$
- Moving source: $f^{\prime}=f v /\left(v-v_{S}\right)$ for a source moving with velocity $v_{S}$
- Both moving: $f^{\prime}=f\left(v+v_{O}\right) /\left(v-v_{S}\right)$
- There are many situations where the frequency and wavelength are not related in a simple, linear way (Sec. 7)
- Wave packets can be represented as a sum of harmonic waves
- The carrier wave (the wave with the average frequency in beats) moves at the phase velocity, $v_{p}=\omega / k$
- The envelope (the slow variation at the difference frequency in beats) moves at the group velocity $v_{g}=d \omega / d k$
- We can also write $v_{g}=v_{p}+k d v_{p} / d k$
- The relationship between angular velocity $\omega$ and wavenumber $k$ is called the dispersion relation
- For a non-dispersive wave, $\omega=c k$


## A The Greek Alphabet

| Name | Lower Case | Upper Case |
| :--- | :---: | :---: |
| alpha | $\alpha$ | A |
| beta | $\beta$ | B |
| gamma | $\gamma$ | $\Gamma$ |
| delta | $\delta$ | $\Delta$ |
| epsilon | $\epsilon$ | E |
| zeta | $\zeta$ | Z |
| eta | $\eta$ | H |
| theta | $\theta$ | $\Theta$ |
| iota | $\iota$ | I |
| kappa | $\kappa$ | K |
| lambda | $\lambda$ | $\Lambda$ |
| mu | $\mu$ | M |
| nu | $\nu$ | N |
| xi | $\xi$ | $\Xi$ |
| omicron | o | O |
| pi | $\pi$ | $\Pi$ |
| rho | $\rho$ | P |
| sigma | $\sigma$ | $\Sigma$ |
| tau | $\tau$ | T |
| upsilon | $v$ | $\Upsilon$ |
| phi | $\phi$ | $\Phi$ |
| chi | $\chi$ | X |
| psi | $\psi$ | $\Psi$ |
| omega | $\omega$ | $\Omega$ |


[^0]:    ${ }^{1}$ The time average of $\cos ^{2} \omega t$ is $\frac{1}{2}$ which can be seen by considering the time average of $\cos ^{2} \omega t+\sin ^{2} \omega t$.

[^1]:    ${ }^{2}$ For a real material, the cross-section will also change a small amount, but we will neglect this.

