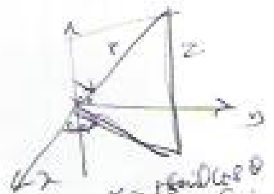
10. Starting from the classical definition of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, show that the quantum mechanical operator for the z-component of angular momentum may be written as

$$\hat{L}_z = -\mathrm{i}\hbar \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right].$$

[3]

In spherical polar coordinates, the corresponding expression is

$$\hat{L}_z = -\mathrm{i}\hbar \frac{\partial}{\partial \phi}.$$



Consider the two states

$$\psi_0 = Az\exp(-r/a), \quad \psi_1 = Ay\exp(-r/a),$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

where a is a constant, $r = \sqrt{x^2+y^2+z^2}$ and A is a constant whose value is chosen to correctly normalize the wavefunctions.

Express ψ_0 and ψ_1 in spherical polar coordinates. [2]

Using either coordinate system, show that ψ_0 is an eigenfunction of $\hat{L}_z$ and find the corresponding eigenvalue. Show also that ψ_1 is not an eigenfunction of $\hat{L}_z$. [3]

Show that ψ_1 is an eigenfunction of the total squared angular momentum

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi} \right]$$

and find the corresponding eigenvalues. What is the corresponding value of the orbital quantum number l? [5]

The completeness postulate implies that the angular part of ψ_1 can be expanded in terms of the spherical harmonics, such that

$$\psi_1(r,\theta,\phi) = \sum_{lm} a_{lm} Y_l^m(\theta,\phi) R(r).$$

By comparing your expression for $\psi_1(r,\theta,\phi)$ with the expressions given below for the spherical harmonics Y_l^m, find the expansion coefficients a_{lm} and determine the form of the function $R(r)$. [3]

Hence determine the possible results of measuring the z-component of the angular momentum in the state ψ_1, and the corresponding probabilities. [4]

[The first few spherical harmonics Y_l^m are:

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta; \quad Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\mathrm{i}\phi); \quad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\exp(-\mathrm{i}\phi).$$

]

$L = p \times$
$P = i\hbar\nabla$