9. It may be shown, from the time-dependent Schrödinger equation, that for an operator $\hat{O}$ that does not explicitly depend on time, the time-derivative of the expectation value is

$$\frac{\mathrm{d}\langle\hat{O}\rangle}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\langle[\hat{O},\hat{H}]\rangle.$$

Consider the case where $\hat{O}$ is the momentum operator

$$\hat{p}_x = -\mathrm{i}\hbar\frac{\partial}{\partial x},$$

and where $\hat{H}$ is the one-dimensional Hamiltonian for a particle of mass m moving in a harmonic potential $V(x) = \frac{1}{2}kx^2$.

(a) By considering the action of $[\hat{p}_x, \hat{H}]$ on a general state, find an expression for the rate of change of $\langle\hat{p}_x\rangle$. [3]

(b) Comment on the relationship to the classical expression for the rate of change of momentum. [2]

(c) To what quantity does the variable $\omega_0 = \sqrt{k/m}$ correspond in the classical system? In terms of ω_0, state the allowed energy eigenvalues according to quantum mechanics. [2]

Making the substitution $y = x\sqrt{m\omega_0/\hbar}$, the time-independent Schrödinger equation for this system is

$$-\frac{d^2\psi}{dy^2} + y^2\,\psi = \frac{2E}{\hbar\omega_0}\psi.$$

The ground state eigenfunction of this equation is $\psi_0 \propto \exp(-y^2/2)$, with eigenvalue $E_0 = \hbar\omega_0/2$.

(d) Verify that $\psi_2 \propto (2y^2 - 1)\exp(-y^2/2)$ is also a solution of the time-independent Schrödinger equation, and that the corresponding energy eigenvalue is $E_2 = 5E_0$. [5]

(e) What are the corresponding solutions, $\Psi_0(y,t)$ and $\Psi_2(y,t)$ of the *time-dependent* Schrödinger equation? [2]

The particle is initially prepared in the state

$$\Psi(y,0) = Ay^2\exp(-y^2/2),$$

where A is a constant chosen in such a way that Ψ is correctly normalized. Express $\Psi(y,0)$ in terms of $\Psi_0(y,0)$ and $\Psi_2(y,0)$. Hence show that the probability density per unit length ρ of finding the particle at the origin $x = 0$ varies with time as [6]

$$\rho = \frac{A^2}{2}\sqrt{m\omega_0/\hbar}\,[1 - \cos(2\omega_0 t)]\,.$$