8. Write down the time-independent Schrödinger equation for a single electron orbiting a nucleus of atomic number $Z$ (assumed to be stationary). [2]

The ground state eigenfunction of this equation, $\psi_{1s}$, and the radially symmetric eigenfunction with next lowest energy, $\psi_{2s}$, are:

$$\psi_{1s} = 2\left(\frac{Z}{a_0}\right)^{3/2}\exp(-Zr/a_0), \quad \psi_{2s} = \frac{1}{\sqrt{2}}\left(\frac{Z}{a_0}\right)^{3/2}\left(1-\frac{Zr}{2a_0}\right)\exp(-Zr/2a_0),$$

where $a_0 = 4\pi\epsilon_0\hbar^2/(m_e e^2)$ is the Bohr radius and the wavefunctions are normalized such that

$$\int_0^\infty r^2|\psi^2|dr = 1.$$

(a) Use the Schrödinger equation to show that the energy in state $\psi_{1s}$ is [4]

$$E_{1s} = -\frac{1}{2}\frac{\hbar^2 Z^2}{m_e a_0^2}.$$

(b) Determine the most probable distance from the nucleus at which the electron is located in the state $\psi_{1s}$. [5]

(c) An atom of tritium ($^3$H) is in its ground state when the nucleus suddenly decays into a helium nucleus ($^3$He), by emission of a fast electron and an electon antineutrino, without perturbing the single extranuclear electron.

Write down the wavefunction of the ground state of tritium. Explain how this wavefunction may be written in terms of the energy eigenfunctions of the resulting $He^+$ ion. [2]

Calculate the probability of finding the resulting $He^+$ ion in the 2s eigenstate. [5]

What are the other energy eigenstates in which the $He^+$ ion could possibly be found, immediately following this nuclear decay? [2]

[ *The radial component of the Laplacian operator is*

$$\nabla_r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

*You may assume the value of the following definite integral*

$$\int_0^\infty r^n\exp(-r/a)dr = n!\,a^{n+1}.$$

]