10. Write down the time-independent Schrödinger equation for an electron in a hydrogen atom, using atomic units and spherical polar coordinates. [2]

In spherical polar coordinates we can separate the variables in the form

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi);$$

show that the angular function Y can be taken to be a spherical harmonic, and (again in atomic units) find the differential equation satisfied by the radial function R for a given value of the angular momentum quantum number l. [7]

The probability that the electron may be found in a spherical shell between distances r and $r+\delta r$ from the nucleus can be written as $r^2|R|^2\delta r$. Write down a suitable normalization condition that should be satisfied by the function R. [2]

Explain what is meant by the expectation value of a physical quantity in quantum mechanics, and deduce a formula for determining the expectation value of a quantity $f(r)$ which depends only on the distance from the nucleus. [3]

For the lowest (1s) state of hydrogen the radial function is $R(r) = C\mathrm{e}^{-r}$, where C is a normalization constant. In this state, find (in atomic units) (i) a suitable value for C and (ii) the expectation value of the potential energy due to the interaction with the nucleus. [6]

[The Laplacian operator can be written in spherical polar coordinates and atomic units as

$$\nabla^2 = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}}{\mathrm{d}r}\right] - \frac{\hat{L}^2}{r^2}.$$

where $\hat{L}^2$ is the square of the orbital angular momentum. You may use without proof the formula for the integrals

$$\int_0^\infty x^n \mathrm{e}^{-\alpha x}\,\mathrm{d}x = \frac{n!}{\alpha^{n+1}}$$

(valid for non-negative integers n and for any real $\alpha > 0$). You may also assume that in atomic units $\hbar = 1$, $m_e = 1$ and $\frac{e^2}{4\pi\epsilon_0} = 1$.]