9. What is meant in quantum mechanics by the statement that a set of functions is *orthonormal*? [3]

The completeness postulate of quantum mechanics states that a general function ψ can be expanded in terms of the eigenfunctions ϕ_n of a Hermitian operator $\hat{O}$ provided it obeys the same boundary conditions. What is the mathematical form of such an expansion? [2]

Assuming that the eigenfunctions are chosen to be orthonormal, obtain a formula for the coefficients in your expansion. [3]

If the eigenvalue of $\hat{O}$ corresponding to ϕ_n is λ_n, what is the probability that a measurement of $\hat{O}$ in the state ψ yields the result λ_n, assuming no other eigenfunction has the same eigenvalue? [2]

A particle of mass m moves in a one-dimensional infinite square well that extends from $x = -a$ to $x = +a$. When the particle is inside the well (i.e. when $-a < x < a$) what is its Hamiltonian? [2]

The particle is prepared in such a well in the normalized state

$$\psi(x) = \begin{cases} C(a^2 - x^2) & \text{when } -a < x < a; \\ 0 & \text{otherwise} \end{cases}$$

where C is a constant such that $|C|^2 = \frac{15}{16a^5}$. Evaluate $\hat{H}\psi$ and hence find the expectation value of the Hamiltonian operator in this state. [4]

Show also that the probability of obtaining the ground-state energy when the energy is measured is $960/\pi^6$. [4]

[You may assume without proof the integral

$$\int_{-\pi/2}^{\pi/2} \left(\frac{\pi^2}{4} - \theta^2\right) \cos\theta \, \mathrm{d}\theta = 4,$$

and that the normalized ground state wave-function for a particle in a one-dimensional infinite square well is $\phi_1(x) = \sqrt{\frac{1}{a}} \cos(\pi x/2a)$.*]*