11. Consider a particle of mass m moving freely in two dimensions but confined within a rectangular box by infinitely high potential walls at $x = \pm a$ and $y = \pm b$.

(a) Write down the time-independent Schrödinger equation satisfied by the wave function $\psi(x, y)$ inside the box (i.e. for $-a \le x \le a; -b \le y \le b$). (You may take the value of the potential energy to be zero in this region.) [4]

(b) Write down the boundary conditions obeyed by ψ at the edges of the box, explaining the reasons for them. [4]

(c) Show that, by writing

$$\psi(x, y) = X(x)Y(y),$$

you can separate the Schrödinger equation into one part depending only on x and one part only on y. Hence show that

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = E_x X,$$

where E_x is a constant whose origin you should explain, and find a corresponding equation satisfied by Y. [5]

(d) Solve the equations for X and Y, subject to the boundary conditions you found in part (b). [4]

(e) Hence show that the lowest energy eigenvalue of the system is

$$\frac{\hbar^2\pi^2}{8m}\left[\frac{1}{a^2} + \frac{1}{b^2}\right]$$

Give an expression for the corresponding ground-state wave function $\psi(x, y)$. (It need not be normalized.) [3]