10. (a) In spherical polar coordinates, the operator $\hat{L}_z$ can be written

$$\hat{L}_z = -\mathrm{i}\hbar\frac{\partial}{\partial\phi}.$$

Show that any function of the azimuthal angle ϕ of the form

$$f_m(\phi) = C\mathrm{e}^{\mathrm{i}m\phi}$$

is an eigenfunction of $\hat{L}_z$; find the corresponding eigenvalue, and explain why m must be an integer. [4]

(b) Now look for eigenfunctions of the operator $\hat{L}^2$, having eigenvalue λ, which are also eigenfunctions of $\hat{L}_z$, in the following way. Try a solution of the form,

$$Y(\theta,\phi) = \Theta(\theta)\mathrm{e}^{\mathrm{i}m\phi},$$

and show that

(i) Y is indeed still an eigenfunction of $\hat{L}_z$; [2]

(ii) The unknown function Θ obeys the equation [4]

$$-\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + m^2\Theta = \frac{\lambda}{\hbar^2}\sin^2\theta\Theta.$$

(c) The solutions to this equation which are finite at $\theta = 0$ and $\theta = \pi$ are the associated Legendre functions

$$\Theta(\theta) = P_l^m(\cos\theta);$$

the eigenvalues are $\lambda = l(l+1)\hbar^2$, where l is a non-negative integer ($l = 0, 1, 2, \ldots$). Using the information about these functions given below, identify the values of l and m, and the corresponding eigenvalues of $\hat{L}_z$ and $\hat{L}^2$, for the following two eigenfunctions (which are not normalized): [6]

$$Y^{(1)} = \sin\theta\mathrm{e}^{\mathrm{i}\phi}; \quad Y^{(2)} = \sin\theta\mathrm{e}^{-\mathrm{i}\phi}.$$

(d) The angular part of a particle's wave-function is given by

$$\psi(\theta,\phi) = A + B\sin\theta\cos\phi.$$

What would be the possible results of measuring the particle's total orbital angular momentum $\hat{L}^2$ and its z-angular momentum $\hat{L}_z$? [4]

[The operator $\hat{L}^2$ can be written in spherical polar coordinates as

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

You may use without proof the expressions

$$P_0^0(\cos\theta) = 1; \quad P_1^{\pm1}(\cos\theta) = \sqrt{1-\cos^2\theta} = \sin\theta; \quad P_1^0(\cos\theta) = \cos\theta$$

for the first few associated Legendre functions.]