8. A particle of mass m moves in a finite one-dimensional rectangular well, located between $x = -a$ and $x = +a$, such that the potential is

$$V(x) = \begin{cases} 0 & (|x| \leq a); \\ V_0 & (|x| > a), \end{cases} \quad \text{with} \quad V_0 > 0.$$

Sketch a graph of the potential $V(x)$. [2]

The time-independent Schrödinger equation inside the well is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = E\psi \quad (|x| \leq a).$$

Write down the general solution to this equation for positive energies E in terms of the wavenumber k. How is k related to E? [4]

What is the Schrödinger equation in the barrier region $|x| > a$? [2]

Assuming the energy E is less than V_0, the general solution in the barrier regions can be written

$$\psi(x) = Ce^{\kappa x} + De^{-\kappa x}, \quad (|x| > a)$$

where C and D are arbitrary constants. Find the value of the constant κ in terms of the energy E, and show that

$$k^2 + \kappa^2 = k_0^2, \quad \text{where} \quad \frac{\hbar^2 k_0^2}{2m} = V_0.$$

[4]

Consider the right-hand barrier region ($x > a$). One of the two terms in the general solution can be ruled out on physical grounds; which is it, and why? [3]

What *two* conditions do the solutions for ψ in the different regions have to satisfy at the edges of the well $x = \pm a$? [2]

In the case of even solutions where $\psi(x) = \psi(-x)$, these two conditions can be shown to require that

$$k\tan(ka) = \kappa = \sqrt{k_0^2 - k^2}.$$

Suppose the particle concerned is an electron. Working in atomic units ($\hbar = m_e = 1$) find the depth V_0 of a well having $a = 1$ unit, given that it possesses an even stationary state with energy $E = 0.125$ units. [3]