9. To which physical quantity does the Hamiltonian operator $\hat{H}$ correspond in quantum mechanics? Write down the form of the Hamiltonian for a particle moving in one dimension in a time-independent potential, and give the equation defining an eigenfunction $\psi_n$ of the Hamiltonian and its corresponding eigenvalue $E_n$. [4]

A solution of the full time-*dependent* Schrödinger equation can be constructed by taking

$$\Psi_n(x,t) = \exp(-\mathrm{i}E_n t/\hbar)\psi_n(x).$$

By substituting into the time-dependent Schrödinger equation, show that any linear combination of two such solutions, in the form $c_1\Psi_1(x,t) + c_2\Psi_2(x,t)$ where $c_1$ and $c_2$ are constants, is also a solution. [4]

In atomic units and spherical polar coordinates, the 1s and 2s stationary-state wave functions of the electron in a hydrogen atom can be written respectively as

$$\psi_{1s}(r,\theta,\phi) = 2\mathrm{e}^{-r}Y_{00}(\theta,\phi); \quad \psi_{2s}(r,\theta,\phi) = \frac{1}{\sqrt{2}}(1-\frac{r}{2})\mathrm{e}^{-r/2}Y_{00}(\theta,\phi).$$

What are the corresponding energies $E_{1s}$ and $E_{2s}$, also in atomic units? Hence write down the corresponding time-dependent solutions $\Psi_{1s}(r,\theta,\phi,t)$ and $\Psi_{2s}(r,\theta,\phi,t)$. [4]

Suppose the electron's wave function at time $t=0$ is

$$\Psi(r,\theta,\phi,t=0) = \frac{1}{3}\psi_{1s}(r,\theta,\phi) + \frac{2\sqrt{2}}{3}\psi_{2s}(r,\theta,\phi).$$

What is the wave function at subsequent times $t$? [4]

Hence show that the probability per unit volume (in atomic units) of finding the electron near the nucleus (at $r=0$) varies with time as

$$|\Psi(r=0,\theta,\phi,t)|^2 = \frac{4}{9\pi}\cos^2\left(\frac{3t}{16}\right).$$

[4]

*[The $(l=0, m=0)$ spherical harmonic is $Y_{00}(\theta,\phi) = 1/\sqrt{4\pi}$. The energy eigenvalue of a hydrogen-atom stationary state having principal quantum number $n$ is $E_n = -1/2n^2$, in atomic units. You may assume that*

$$\hbar = 1; \quad m_e = 1; \quad \frac{e^2}{4\pi\epsilon_0} = 1$$

*in atomic units.]*