Problem Set 5

1. Consider two media with refractive index $n_1 = 1$ and $n_2 = \sqrt{5/2} = 1.581$ with a plane surface as boundary. Light is incident from a point A in medium 1 which is 10cm from the boundary and travels to point B in medium 2 which is 10cm the other side of the boundary. Points A and B are hence separated by a distance of 20cm along the axis perpendicular to the boundary. They are also separated by 15cm in a direction parallel to the boundary. Show that the straight-line distance between the points is 25cm. [1]

If the light does travel in a straight line between the points what is the optical path length? How does this compare to the true path which satisfies Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (where θ_1 and θ_2 are the angles between the light ray and the perpendicular to the boundary in each medium), where $\sin\theta_1 = 1/\sqrt{2}$ and $\sin\theta_2 = 1/\sqrt{5}$? [6]

2. A light ray is incident from a medium of refractive index n_1 at an angle 40^0 to the normal to a plane surface which forms the boundary to medium 2, with index of refraction n_2. It continues at angle 20^0 to the normal and then strikes another boundary, parallel to the first. It continues through this boundary reaching a vacuum, where it now travels at an angle 60^0 to the normal.

Find the values of n_1 and n_2. [5]

Explain what happens if the incident beam is incident at an angle of 60^0 to the normal. [2]

3. We have a system of three consecutive polarising sheets. The axes of second and the third are at angles θ and 90^0 respectively to that of the first. Unpolarised light with intensity I_0 is normally incident on the first sheet.

a. Derive an expression for the intensity of light transmitted through the system as a function of I_0 and θ, showing it is $I = 1/8\, I_0 \sin^2 2\theta$. [4]

b. For what value of θ does the maximum transmission occur? [3]

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Problem Set 5 - Answers.

1. In the below generic figure for refraction then $a = b = 10$cm and $d = 15$cm. Hence, if the light travels in a straight line then the path is obtained using Pythagoras' theorem where the two sides of the right-angled triangle have length 15cm and 20cm. Therefore the distance is

$$L = \sqrt{15^2 + 20^2}\text{cm} = \sqrt{625}\text{cm} = 25\text{cm}.$$

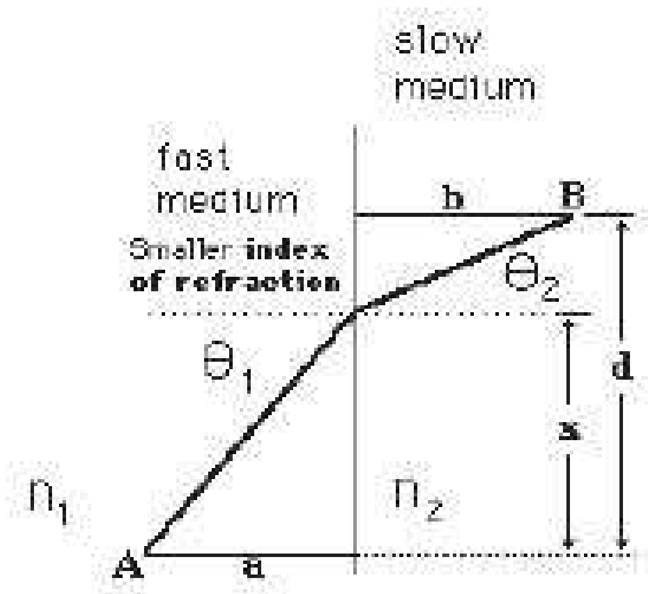

Figure 1:

In this case 12.5cm of the path is in each medium so the path length is

$$n_1 \times 12.5\text{cm} + n_2 \times 12.5\text{cm} = 12.5\text{cm} + 1.581 \times 12.5\text{cm} = 32.26\text{cm}.$$

If instead the path is the one which satisfies Snell's law the distance in medium 1 is $x = 10\text{cm} \times \tan\theta_1 = 10$cm, while the distance in medium 2 is $x = 10\text{cm} \times \tan\theta_2 = 10 \times 1/2 =$5cm. Hence, in this case the path distance is

$$n_1 \times \sqrt{10^2 + 10^2}\text{cm} + n_2 \times \sqrt{10^2 + 5^2}\text{cm} = 14.14\text{cm} + 1.581 \times 11.18\text{cm} = 31.82\text{cm}.$$

So the true path has the shorter optical path length, consistent with Fermat's theorem.

2. At each boundary we have the application of Snell's law, i.e.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \;\text{ and }\; n_2 \sin\theta_2 = \sin\theta_{vac}.$$

From the second condition

$$n_2 = \sin\theta_{vac}/\sin\theta_2 = \sin 60^0/\sin 20^0 = 2.53.$$

Inputting into

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \rightarrow n_1 = 2.53 \sin 20^0/\sin 40^0 = 1.35.$$

If the incident beam is incident at an angle of 60^0 to the normal

$$\sin\theta_2 = (1.35/2.53)\sin 60^0 = 0.462 \rightarrow \sin\theta_2 = 27.5^0.$$

This then leads to

$$\sin\theta_{vac} = 2.53 \sin\theta_2 = 1.17$$

which is greater than 1. So there is no real solution for θ_{vac} and we have total internal reflection at the boundary of medium 2 and the vacuum.

3. a. From the first polariser we obtain linearly polarised from unpolarised light. Therefore intensity is average over of $I_0 \cos^2\theta$ over all angles, i.e.

$$I_1 = \tfrac{1}{2} I_0.$$

Using the Law Of Malus

$$I_2 = I_1 \cos^2\theta = \tfrac{1}{2} I_0 \cos^2\theta.$$

The angle ψ between second and third polarsier is $\psi = 90^0 - \theta$ so

$$I_3 = I_2 \cos^2(\psi) = I_2 \cos^2(90^0 - \theta) = I_2 \sin^2\theta = \tfrac{1}{2} I_0 \cos^2\theta \sin^2\theta.$$

But $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, so

$$I_3 = 1/8 I_0 \sin^2 2\theta.$$

b. To maximise intensity we must find $dI_3/d\theta = 0$, which gives

$$\begin{aligned}
\frac{dI}{d\theta} &= 1/8 I_0 \times 4 \sin 2\theta \cos 2\theta \\
&= \tfrac{1}{2} I_0 \sin 2\theta \cos 2\theta \\
&= 1/4 I_0 \sin 4\theta.
\end{aligned}$$

This is zero for $4\theta = \pi$, i.e. $\theta = \pi/4$.

Problem Set 6

1. The precise expression for the reflection coefficients for the components of the light parallel to the boundary and perpendicular are

$$r_{\parallel} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}, \qquad r_{\perp} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}.$$

Using Snell's law to eliminate θ_t show that $r_{\parallel} = 0$ if [3]

$$\frac{n_t^2}{n_i^2}\cos^2\theta_i = 1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i$$

Using this result show that there is an angle $\theta_B = \tan^{-1}(n_t/n_i)$ (Brewster's angle) at which $r_{\parallel} = 0$, and at which the reflected light is polarised (you will find it useful to use the relationship $1 = \sin^2\theta + \cos^2\theta$). [4]

2. Two flat rectangular glass sheets touch at one end and are separated by a hair 100mm from, and parallel to, that edge to form a wedged shaped gap. When illuminated at near-normal incidence with light with a wavelength of 500nm fringes are observed in reflection with a spacing of 1mm. Calculate the thickness of the hair. [5]

The gap is now filled with oil of refractive index $n = 1.5$ How does the separation of the fringes change? [2]

3. In air a lens with one convex curved surface and one flat surface lies on a flat horizontal reflecting surface with curved side downwards. The radius of curvature for the lens is 20cm. The flat surface of the lens is illuminated from above with sodium light with wavelength $\lambda_0 = 589.29$nm. The light strikes the flat surface of the lens at right angles to the surface, i.e. we have set up the apparatus for Newton's rings. The refractive index of air may be assumed to be $n = 1$. The interference pattern is observed and then the gap between the curved surface of the lens and the surface it rests on is filled with a liquid with refractive index $n = 1.461$. What is the ratio of the radius of the 13_{th} dark band before introducing the liquid to after introducing it? [5]

What is the radius of the 13_{th} dark band in the second case? [2]

Extra questions not for Assessment.

4. Right-circularly polarised light travelling in the z direction may be generated by first passing unpolarised light through a linear polariser with axis along the line $x = y$ and then introducing a quarter wave plate which retards the component along the y axis by $\pi/2$ compared so that along the x axis. For left-circularly polarised light the polariser is the same but the plate is such that the component along the y axis is advanced by $\pi/2$ compared to that along the x axis.

A receptor for each polarisation can be obtained by rotating the generator by π about the y-axis so that the effect of the quarter wave plate remains the same, but the axis of the linear polariser is along the line $x = -y$. Show that this does result in all the light from the right-circular generator being transmitted by the right-circular receiver. [3]

Explain what happens when light from the right-circular generator is incident on the left-circular receiver. [3]

A pair of glasses used for viewing 3D films consists of one lens which is a right-circular receiver (the right-hand lens) and one which is a left-circular receiver (the left-hand lens), i.e. for each the light first goes through the wave plate, which has an opposite effect for each lens, then through the linear polariser, which is the same for each lens. In this case we assume the linear polariser is along the x axis and the quarter wave plate retards or advances along the line $x = y$. We label the face where light usually enters as A and where it leaves to go to the eye as B. Extend the argument in the last part of the question to explain the features of the glasses viewed in the lectures, i.e.

a. When face A of one set is adjacent to face B of the other (i.e. one set is simply on top of the other) light is transmitted whatever the relative orientation of the glasses, or whichever lens is adjacent to the other. [2]

b. When face B of one pair is adjacent to face B of the other there is no dependence on whether the right-hand lens of one pair is next to the right-hand or the left-hand lens of the other pair, and light is transmitted if both pairs have the same orientation, but is not if they are tilted at right angles to each other whatever the orientation. [2]

c. When face A of one pair is adjacent to face A of the other (i.e. glasses face each other), if one looks through the right-hand lens of each pair light is transmitted irrespective of orientation, while if one looks through the left-hand lens of one pair and the right-hand lens of the other light is not transmitted. [2]

5. When using the glasses in question 3 we also notice that in situation a, although light was transmitted independent of the relative orientation of the glasses, it was orange/yellow when they had the same orientation and blue when they were at right angles. Explain why this happens. For the even more ambitious, explain why in c if one looks through the right-hand lens of one pair and the left-hand lens of the other there is no transmission when they are at right angles but a very small amount of purple light gets through when they are parallel.

Problem Set 6 - Answers.

1. $r_{\|} = 0$ if $n_t \cos\theta_i = n_i \cos\theta_t$ But from Snell's law $\sin\theta_t = (n_i/n_t)\sin\theta_i$. Therefore, $r_{\|} = 0$ if

$$n_t^2 \cos^2\theta_i = n_i^2(1 - \sin^2\theta_t),$$

and using Snell's law this becomes

$$n_t^2 \cos^2\theta_i = n_i^2\left(1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i\right).$$

And so dividing through by n_i^2,

$$\frac{n_t^2}{n_i^2}\cos^2\theta_i = 1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i. \qquad [3]$$

Using the identity $1 = \sin^2\theta + \cos^2\theta$ in the above

$$\frac{n_t^2}{n_i^2}\cos^2\theta_i = \cos^2\theta_i + \sin^2\theta_i - \frac{n_i^2}{n_t^2}\sin^2\theta_i.$$

Putting all common terms on the same side

$$\cos^2\theta_i\left(\frac{n_t^2}{n_i^2} - 1\right) = \sin^2\theta_i\left(1 - \frac{n_i^2}{n_t^2}\right).$$

Therefore

$$\cos^2\theta_i \frac{n_t^2 - n_i^2}{n_i^2} = \sin^2\theta_i \frac{n_t^2 - n_i^2}{n_t^2},$$

which quickly leads to

$$\frac{\cos^2\theta_i}{n_i^2} = \frac{\sin^2\theta_i}{n_t^2} \quad \rightarrow \quad \frac{\sin^2\theta_i}{\cos^2\theta_i} \equiv \tan^2\theta_i = \frac{n_t^2}{n_i^2}.$$

taking the square root of both sides then $r_{\|} = 0$ for the angle $\theta_p = \tan^{-1}(n_t/n_i)$ (Brewster's angle). [4]

2. From the expression in lectures we get constructive interference when

$$x = \frac{\lambda(p + \frac{1}{2})}{2\alpha},$$

where α is the wedge angle and p is an integer. So the distance between two fringes is when p increases by 1, i.e.

$$\Delta x = \frac{\lambda}{2\alpha} \quad \rightarrow \quad \alpha = \frac{\lambda}{2\Delta x}.$$

Here $\lambda = 500$nm and $\Delta x = 1$mm. Therefore $\alpha = 2.5 \times 10^{-4}$ radians. But if d is the diameter of the hair,

$$\alpha = d/100\text{mm},$$

and so $d = 0.025$mm. [3]

If the gap is now filled with oil of refractive index $n = 1.5$ the optical path difference of the light travelling through the gap is increased by a factor of 1.5, i.e. the separation between fringes is given by

$$1.5\Delta x = \frac{\lambda}{2\alpha}.$$

So the separation of the fringes reduces to $1\text{mm}/1.5 = 0.666$mm. [2]

3. The radius of the p_{th} bright band is given from the lecture notes as $r^2 = (p + \frac{1}{2})R\lambda/n$, where R is the lens curvature. It is $p + \frac{1}{2}$ because the phase shift at one boundary means constructive interference needs a half-integer number of wavelengths path difference. Dark bands require destructive interference and an integer number of wavelengths path difference so $r^2 = pR\lambda/n$. Hence,

$$r_p = \sqrt{p\lambda R/n}.$$

Denoting the radius of the band in air by r_p^a and that in the liquid by r_p^l and similarly for the refractive indices then we find that

$$\frac{r_p^a}{r_p^l} = \sqrt{\frac{(p\lambda R/n^a)}{(p\lambda R/n^l)}}.$$

Using $n^a = 1$ then the cancellation of other terms leads to

$$\frac{r_p^a}{r_p^l} = \sqrt{n^l},$$

independent of the order of the fringe. In this case this is

$$\frac{r_p^a}{r_p^l} = \sqrt{1.461} = 1.209,$$

The radius of the 13th dark band in the second case is

$$r_{13}^l = \sqrt{13 \times 589.29 \times 10^{-9} \times 0.2/1.461} = 1.02\text{mm}.$$

Extra Questions not for Assessment.

4. The right-circularly polarised light has equal components of the field vector along the x and y axes but the component along the y axis is retarded by $\pi/2$. If it travels through the quarter wave plate again the phase of the light with field along the y axis is delayed by a further $\pi/2$ i.e. by a total of π. But a phase change of π simply reverses the direction of the field. Thus the part of the field polarised along the y axis is now polarised along the negative y axis. The component polarised along the x axis is unchanged, so after passing through

the wave plate the second time the light is linearly polarised along the line $x = -y$. But this is the axis of polarisation for the linear polariser in the receiver so all the light from the right-circular generator is transmitted by the right-circular receiver. [3]

The right-circularly polarised light has equal components of the field vector along the x and y axes but the component along the y axis is retarded by $\pi/2$. Passing through the wave plate for the left circular polariser this retardation is reversed and the field components along the x and y axes are back in phase. Thus we have light linearly polarised along the line $x = y$. But the axis of polarisation for the linear polariser in the receiver is the line $x = -y$ which is perpendicular, so no light is transmitted when light from the right-circular generator is incident on the left-circular receiver. [3]

a. When face A of one set is adjacent to face B of the other (i.e. one set is simply on top of the other) unpolarised light enters the top pair of glasses, passes through the wave plate, and hence remains unpolarised. It then passes through the polariser so becomes linearly polarised along the x axis. Passing through either wave plate when entering the second pair of glasses produces either right or left circularly polarised light. This then hits a second polariser with axis along the x axis. Since circularly polarised light is rotating, on average half of it is transmitted by a linear polariser along any axis in the plane of its rotation, so half the light gets through regardless of the orientation of the linear polariser. [2]

b. When face B of one pair is adjacent to face B of the other the linear polariser of the two pairs are adjacent with no wave plate between them. The axis of this is the same in each case, it is the wave plates which have opposite effects in each lens. Hence, this is just like having two equal linear polarisers next to each other, whichever lens is next to the other. If they have the same orientation all light is transmitted. If the are at right angles none is. [2]

c. When face A of one pair is adjacent to face A of the other (i.e. glasses face each other), the light entering one is circularly polarised, i.e. the light enters from the side where the lens acts as a generator of circularly polarised light. However, the second lens is still acting as a receiver. Hence, if light enters the right-hand lens it will be right-circularly polarised. It will then be transmitted fully by the second right-hand lens acting as a receiver. This does not depend on orientation since there is no preferred axis for circularly polarised light. If light enters the right-hand lens, becoming right-circularly polarised and then this enters the left-hand lens, the latter is the receiver for left-circularly polarised light and nothing is transmitted. Clearly the argument is the same if the incoming light enters the left-hand lens. It is fully transmitted by the second left-hand lens, but not transmitted if the second lens is the right-hand lens. [2]

5. Here we have to take into account that the generation of circularly polarised light relies on introducing a $\pi/2$ phase difference. This means there is a quarter wavelength difference in light transmitted along the two perpendicular directions. This can only be exactly true for one wavelength, which will be somewhere near the middle of the visible range i.e $\lambda \approx 450$nm.

In part a the light transmitted by the first pair of glasses is linearly polarised along the line x axis. Passing through the quarter wave plate the light at 450nm will be circularly polarised. That at the blue end, i.e. shorter wavelength will have one component with a phase shift $> \pi/2$, and so this component will be nearer to being π out of phase, i.e. the light will be nearer to linear polarisation at right angles to the initial polarisation. That near the red end, i.e. longer wavelength will have a component with phase shift $< \pi/2$ so will have a degree of linear polarisation of the same orientation as that transmitted. Hence, if the two lenses have the same orientation, so the two linear polarisers are along the same axis, there will be preferential transmission at the redder end compared to the blue. If the second lens is at right angles to the first the axis of transmission of the second polariser is now at right angles to the first and there will be preferential transmission at the blue end of the spectrum.

In case c only the light at 450nm will be generated by the first lens with exact circular polarisation. That at lower wavelength will have a component with phase shift $> \pi/2$ and that at higher wavelength will have a component with phase shift $< \pi/2$. Consider the first lens as the right-hand lens. The wave plate in this lens delays the field component along the line $x = y$. If the second lens is the left-hand lens and is parallel to the first the wave plate in this delays the component along the line $x = -y$ axis, but rotation by 90^0 about the y axis to get the lenses facing each other swaps x for $-x$ axes so the delay is actually again along the line $x = y$. This means that the light with $\lambda = 450$nm has had the phase of the field along the line $x = y$ changed by π, so its direction is reversed and on reaching the second polariser there is a component along the line $x = -y$, i.e. $\propto \mathbf{i} - \mathbf{j}$ and along the negative part of the line $x = y$, i.e. $\propto -\mathbf{i} - \mathbf{j}$. Hence, the resultant is along the (negative) y axis, and is perpendicular to the linear polariser so no light is transmitted. However, for λ not exactly equal to 450nm the phase shift is not exactly π. So the component along $x = y$ is not completely reversed and the resultant is not exactly along the y axis. Hence, at the red and blue end of the spectrum some light gets through giving a purple transmission.

If the second lens is perpendicular to the first, the axis where the phase shift is introduced is further rotated by another 90^0. Hence, one lens delays along the line $x = y$ and the other along the line $x = -y$. Hence, in total both field components are delayed by the same amount and the light arriving at the second linear polariser has both components back in phase and the polarisation is the same as transmitted by the first, i.e. along the x axis. However, if the second lens is oriented perpendicular to the first the linear polarisation of the second lens is along the y axis, and no light is transmitted independent of wavelength.

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R S Thorne

Problem Set 7

1.a. A Michelson interferometer uses light with a wavelength 600nm. How many fringe shifts are observed at the centre of the screen if the movable mirror is moved by 0.1cm? [3]

b. A container 10cm long with flat parallel windows at each end and filled with air is placed in one arm of the Michelson interferometer and a fringe pattern is set up for the same light. The air is pumped out of the container and the pattern shifts by 100 fringes. What is the refractive index of the air? [4]

2.a. The two sodium "D-lines" are at 589.00nm and 589.59nm. What reflectance R is needed for a Fabry-Perot interferometer to resolve these lines in the tenth order? [3]

b. What is the minimum spacing of the plates t in order for this to be possible? [2]

c. If R and t each just satisfy the conditions in parts a. and b. and R is fixed, by how much would we have to increase t to resolve the red lines in hydrogen and deuterium at 656.3nm and 656.1nm respectively? [2]

3. As shown in the lectures the pattern formed on a screen when light is incident on two narrow slits distance d apart is $4I_0\cos^2(\delta/2)$, where I_0 is the intensity arising from one slit and $\delta = (2\pi/\lambda)d\sin\theta$, where θ is the angle between the normal to the separation of the slits and the point on the screen.

a. Explain what happens to the pattern if immediately beyond one slit we introduce a plate of refractive index $n = 2$ and thickness such that the light takes half a wavelength to travel through the plate. (Ignore refraction effects due to the plate.) [2]

b. Alternatively the transmission of one slit is limited to that the amplitude of light is halved compared to the other slit. Using the definition for the visibility of fringes, $V = (I_{\max} - I_{\min})/(I_{\max} + I_{\min})$ find the value of V in this case. [5]

Not for Assessment

4. Each successive ray in the Fabry Perot etalon has amplitude $R\exp(i(2\pi/\lambda)2t\cos\theta)$ compared to the last. This means the total amplitude is proportional to

$$1 + R\exp(i(2\pi/\lambda)2t\cos\theta) + (R\exp(i(2\pi/\lambda)2t\cos\theta))^2 + \cdots \tag{1}$$

where the sum goes on for an effectively infinite number of terms. Remembering that $R < 1$ sum this geometric progression and then take the modulus squared to show that the intensity

$$\propto \frac{1}{1 + R^2 - 2R\cos((2\pi/\lambda)2t\cos\theta)}. \tag{2}$$

Show that this is proportional to

$$\frac{1}{1 + (4R)/(1-R)^2\sin^2((2\pi/\lambda)t\cos\theta)}. \tag{3}$$

Problem Set 7 - Answers.

1.a. A Michelson interferometer produces a new fringe each time the distance d increases by $\lambda/2$, which causes the path length to increase by λ. Hence,

$$\Delta p = \frac{2\Delta d}{\lambda} = \frac{2 \times 0.1 \times 10^{-2}}{600 \times 10^{-9}} = 3,333. \qquad [3]$$

b. The change in optical path length between the two situations is the change of refractive index multiplied by twice the length of the container (since the beam travels through it in both directions). If this results in a change of p fringes then

$$p\lambda = 2(n_{\rm air} - 1)L \rightarrow n_{\rm air} - 1 = p\lambda/(2L).$$

In this case $L = 10$cm, $\lambda = 600$nm, and $p = 100$ so we obtain

$$n_{\rm air} - 1 = \frac{100 \times 6 \times 10^{-7}}{2 \times 0.1} = 0.0003.$$

So the refractive index of air is 1.0003 [4].

2.a. The chromatic resolution of the interferometer is given by

$$\lambda/\Delta\lambda = m\pi\sqrt{F} = m\pi\left(\frac{4R}{(1-R)^2}\right)^{\frac{1}{2}}.$$

In this case

$$\frac{4R}{(1-R)^2} = (\frac{589}{0.59 \times 31.4})^2 = 1011.$$

This leads to

$$4R = 1011 - 2022R + 1011R^2 \quad \rightarrow \quad 0 = 1 - 2.004R + R^2.$$

Solving the quadratic equation, and taking the solution for $R < 1$ we obtain $R = 0.94$. [3]

b. We obtain bright fringes when $2t\cos\theta = m\lambda$, so if $m = 10$ this can be obtained for the minimum t if $\cos\theta = 1$, i.e. $\theta = 0$. In this case

$$2t = 10\lambda \quad \rightarrow \quad t = 5\lambda = 2.95 \times 10^{-6}\text{m}. \qquad [2]$$

c. For these lines

$$\lambda/\Delta\lambda = 656/0.2 = 3280.$$

This is more than 3.2 times bigger than the previous resolution, so if R and hence F remain the same we must go to $m = 33$ to obtain resolution. At $\theta = 0$ this means

$$t = 33\lambda/2 = 16.5 \times 656 \times 10{-}9 = 1.08 \times 10^{-5}\text{m}. \qquad [2]$$

3.a If immediately beyond one slit we introduce a plate of refractive index $n = 2$ and thickness such that the light takes half a wavelength to travel through the plate, the light from this slit travels half a wavelength while the light from the other slit travels a quarter of a wavelength. This introduces a shift of pattern by $\pi/2$, so where we had peaks we are now at half maximum height. [2]

b. When we add waves of unequal amplitude, e.g. amplitudes A and B we obtain light intensity of

$$I = A^2 + B^2 + 2AB\cos\delta.$$

So $I_{\text{max}} = (A+B)^2$ when $\cos\delta = 1$ and $I_{\text{min}} = (A-B)^2$ when $\cos\delta = -1$. So if $A = 1$ and $B = \frac{1}{2}$, $I_{\text{max}} = 9/4$ and $I_{\text{min}} = 1/4$. Therefore

$$V = (9/4 - 1/4)/(9/4 + 1/4),$$

i.e. $V = 4/5$. [5]

Not for Assessment

4. If we have a geometric progression

$$1 + x + x^2 + \cdots + x^N,$$

where $|x| < 1$ and $N \to \infty$ the sum is $1/(1-x)$. In this case that means the intensity

$$\propto \frac{1}{1 - R\exp(i(2\pi/\lambda)2t\cos\theta)},$$

and so the intensity

$$\propto \frac{1}{1 - Re^{(i(2\pi/\lambda)2t\cos\theta)}} \times \frac{1}{1 - Re^{(-i(2\pi/\lambda)2t\cos\theta)}} = \frac{1}{1 + R^2 - 2R\cos((2\pi/\lambda)2t\cos\theta)}.$$

This can be rewritten as

$$\frac{1}{(1-R)^2 + 2R(1 - \cos((2\pi/\lambda)2t\cos\theta))},$$

and using $(1 - \cos(2x)) = 2\sin^2 x$ this is

$$\frac{1}{(1-R)^2 + 4R(\sin^2((2\pi/\lambda)t\cos\theta))}.$$

Dividing through by $(1-R)^2$ to make the normalisation of the maximum value equal to 1 we obtain

$$\frac{1}{1 + (4R)/(1-R)^2\sin^2((2\pi/\lambda)t\cos\theta)}.$$

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Problem Set 8

1. A pointillistic painting contains many small coloured dots. When the viewer is very close to the canvas the individual dots and their colours can be distinguished. At normal viewing distance the dots are not distinguishable and blend.

a. Assume that the average spacing of the centre of the dots is $d = 2.0$mm. Also assume that the diameter D of the pupil of your eye is $D = 2.0$mm. and that the least angular separation between dots you can resolve is given by Rayleigh's criterion. As you move away from the picture which same-coloured adjacent dots become indistinguishable first, red ($\lambda = 700$nm) or blue ($\lambda = 400$nm). [2]

b. At what distance do you cease being able to resolve any dots on the painting? [3]

c. If the artist wished all dots to become resolvable at the same distance and blue dots are separated by $d = 2.0$mm how much should the red dots be separated by? [3]

2. The pattern from a diffraction grating is shown below in figure 1 where the x-axis is in units of $\pi d \sin\theta/\lambda$, and the height of the principal maximum is in units of N^2. The slits have been assumed to be extremely narrow.

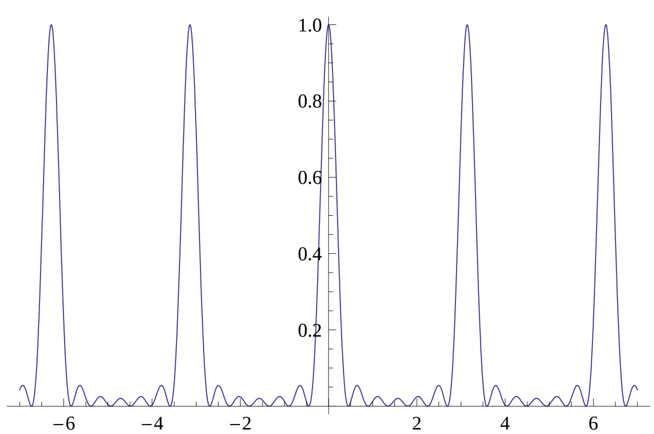

Figure 1:

a. State how many slits N there are in the array and explain your answer. [3]

b. Work out the height of the second subsidiary maximum (assuming it is exactly where $\sin^2(N\pi d \sin\theta/\lambda)$ would have a maximum). [3]
c. Show that for the particular case $N = 2$ the diffraction pattern for the grating $\sin^2(N\pi d \sin\theta/\lambda)/\sin^2(\pi d \sin\theta/\lambda)$ is the same as that for the double slit, i.e. $4\cos^2(\pi d \sin\theta/\lambda)$. [3]

3. a. A concave mirror has a focal length of 20cm. What is the position of the image formed if it is inverted and 4 times smaller than the object? [3]

b. A convex mirror has a focal length of -20cm. What is the position of the image formed if it is upright and 4 times smaller than the object? [3]

4. An 2 cm object is 20cm to the left of a converging lens with a focal length of 10cm. A diverging lens of focal length -5cm is 30cm to the right of the first lens.

a. Calculate the position of the final image. [5]

b. Calculate the overall lateral magnification M of the two-lens system and the height of the final image. [2]

c. Is the final image upright or inverted? [1]

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R S Thorne

Problem Set 8

1. a. We can resolve angles down to $\theta = 1.22\lambda/D$. Hence, the smaller the wavelength the smaller the angle, and as we move away the red spots cease to be distinguishable first. [2]

b. For $\lambda = 400$nm then the minimum angle is

$$\theta = 1.22\frac{4 \times 10^{-7}}{2 \times 10^{-3}} = 2.44 \times 10^{-4}. \tag{1}$$

For small angles $\theta = d/L$ where d is separation and L is distance away. So

$$L = \frac{2 \times 10^{-3}\text{m}}{2.44 \times 10^{-4}} = 8.2\text{m}. \qquad [3] \tag{2}$$

c. Since $\theta_{\text{min}} \propto \lambda$ and $L = d/\theta_{\text{min}}$, for the distance to be the same for all colours the separation should be proportional to λ. So if it is $d = 2.0$mm for blue it is

$$d = 2.0\text{mm}(\lambda_{\text{red}})/(\lambda_{\text{blue}}) = 3.5\text{mm} \tag{3}$$

for red light. [3]

2. a. The diffraction pattern intensity is

$$\sin^2(N\pi d \sin\theta/\lambda)/\sin^2(\pi d \sin\theta/\lambda), \tag{4}$$

and has primary maxima at $d\sin\theta = m\lambda$ for integer m, and has zeros for $d\sin\theta = n/N$ for integer n except at the primary maxima where $n/N = m$. This leads to $N-1$ minima between the primary maxima. In this case there are 6 minima, so $N = 7$. [3]

b. The second subsidiary maximum will be very near to $7\pi d\sin\theta/\lambda = 5\pi/2$. This means $\sin^2(7\pi d\sin\theta/\lambda) = 1$ but

$$\sin^2(\pi d\sin\theta/\lambda) = \sin^2(5/14) = 0.90^2. \tag{5}$$

So the intensity will be $1/0.90^2 = 1.23$ compared to $N^2 = 49$ at the primary maximum. [3]

c. We know that $\sin(2x) = 2\sin x \cos x$, so

$$\frac{\sin^2(2\pi d\sin\theta/\lambda)}{\sin^2(\pi d\sin\theta/\lambda)} = \frac{(2\sin(\pi d\sin\theta/\lambda)\cos(\pi d\sin\theta/\lambda))^2}{\sin^2(\pi d\sin\theta/\lambda)} = 4\cos^2(\pi d\sin\theta/\lambda). \quad [3] \tag{6}$$

3. a. We use the mirror equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}, \tag{7}$$

where p is object distance, q is object distance and f is the focal length. The magnification $M = -q/p = -1/4$, so we have that $p = 4q$. This gives

$$\frac{1}{4q} + \frac{1}{q} = \frac{1}{f} \quad \rightarrow \frac{4q}{5} = f, \tag{8}$$

resulting in $q = 1.25 \times 20\text{cm} = 25\text{cm}$. [3]

b. The magnification $M = -q/p = 1/4$, so we have that $p = -4q$. This gives

$$-\frac{1}{4q} + \frac{1}{q} = \frac{1}{f} \quad \rightarrow \frac{4q}{3} = f, \tag{9}$$

resulting in $q = 0.75 \times (-20\text{cm}) = -15\text{cm}$. [3]

4. a. Let the image from the first lens be the object for the second lens. So $p_1 = 20\text{cm}$ and $f_1 = 10\text{cm}$ and

$$\frac{1}{q_1} = \frac{1}{f_1} - \frac{1}{p_1} \quad \rightarrow \quad q_1 = \frac{f_1 p_1}{p_1 - f_1} = \frac{10 \times 20}{20 - 10}\text{cm} = 20\text{cm}, \tag{10}$$

and magnification $m_1 = -q_1/p_1 = -20/20 = -1$.

For the second lens, $p_2 = 30\text{cm} - 20\text{cm} = 10\text{cm}$. Applying the lens equation again gives

$$q_2 = \frac{f_2 p_2}{p_2 - f_2} = \frac{-5 \times 10}{10 + 5}\text{cm} = -3.33\text{cm}. \tag{11}$$

and $m_2 = -(-3.3)/10 = 0.33$. So, the image is 3.33cm to the left of the second lens. [5]

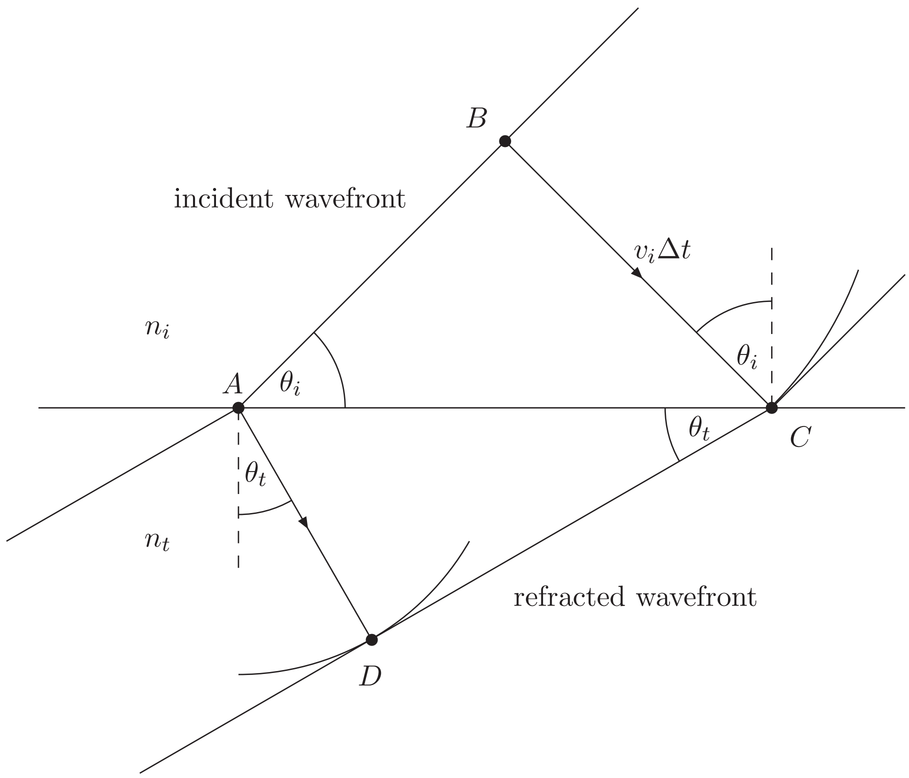
b. Overall lateral magnification $M = m_1 \times m_2 = -1 \times 0.33 = -0.33$, so the height of the final image is $h_f = h \times |M| = 2 \times 0.33 = 0.66\text{cm}$. [2]

c. Since $M < 0$ the final image is inverted. [1]

PHAS1224 Waves, Optics and Acoustics

Problem Solving Tutorial III

1. The diagram showing the projection of a wavefront using Huygens' principle at the boundary of two media with refractive indices n_i and n_t is shown below.



What is the length of the line AD on the above figure? By relating the lengths of AD and BC, and using the fact that ABC and ADC are right-angled triangles, derive Snell's law, i.e. $n_i \sin\theta_i = n_t \sin\theta_t$.

2. The electric field for a light wave takes the form

$$\mathbf{E} = E_0(\cos(kz - \omega t) + \sin(kz - \omega t))\mathbf{i} + E_0(\cos(kz - \omega t) - \sin(kz - \omega t))\mathbf{j}. \quad (1)$$

By choosing a more convenient basis than the unit vectors $\mathbf{i}$ and $\mathbf{j}$ rewrite this expression for the field so that the form of polarisation is more clear. What type of polarised light is it? What are the maximum and minimum amplitudes, and in which direction would an observer at $z = 0$ see the light vector rotate as time increases?

3. If we have light incident from a medium with one refractive index n_i towards a flat boundary with another medium of refractive index n_t, at angle θ_i to the normal the equation for the reflection coefficient for the light with field polarised perpendicular to the plane of incidence is

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}. \quad (2)$$

Using Snell's law to eliminate dependence on refractive indices show that this is equivalent to

$$r_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \tag{3}$$

4. The reflection coefficient for light with field polarised parallel to the plane of incidence is $r_\parallel$. This is zero at at Brewster's angle $\theta_P = \tan^{-1}(n_t/n_i)$. Find this angle for light incident on glass ($n = 1.4$) from air ($n = 1$), and vice versa, and in each case find the value of $r_\perp$ at this angle. In the latter case find also the angle at which total internal reflection takes place, and compare to Brewster's angle.

5. Monochromatic blue light with $\lambda_0 = 487.99$nm is normally incident on a soap film with $n = 1.555$, and thickness 1.648×10^{-6}m. Light is reflected from the front surface, and then the back surface of the soap film, producing two waves travelling backwards from the original direction of propagation. Find the difference in the optical path length travelled by the two waves. What phase difference does this correspond to? Is then any additional phase shift associated with either of the reflections? Taking any extra shift into account state whether the two reflected components add constructively or destructively?

PHAS1224 Waves, Optics and Acoustics

Problem Solving Tutorial III

1. In time interval Δt the wavefront in the medium with refractive index n_t travels distance $v_t \Delta t = c/n_t \Delta t$. Hence, the length of AD is $v_t \Delta t$.

We can see from the geometry that

$$\sin\theta_i = v_i \Delta t / AC \qquad \text{and} \qquad \sin\theta_t = v_t \Delta t / AC. \tag{1}$$

Therefore,

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{v_t}{v_i} = \frac{c/n_t}{c/n_i} = \frac{n_i}{n_t}. \tag{2}$$

Hence,

$$n_i \sin\theta_i = n_t \sin\theta_t. \tag{3}$$

2. Since there are terms involving both $\cos(kz - \omega t)$ and $\sin(kz - \omega t)$ then this will be related to circularly or elliptically polarised light. It is easiest to analyse the form of the wave by grouping together terms where the dependence on phase is the same, i.e. grouping $\cos(kz - \omega t)$ terms together and $\sin(kz - \omega t)$ terms together. This results in

$$\mathbf{E} = E_0 \cos(kz - \omega t)(\mathbf{i} + \mathbf{j}) + E_0 \sin(kz - \omega t)(\mathbf{i} - \mathbf{j}). \tag{4}$$

This means we can think of the wave as one linearly polarised wave along the direction given by $\mathbf{i} + \mathbf{j}$ and another linearly polarised wave along the direction given by $\mathbf{i} - \mathbf{j}$ with equal amplitude, but with phase $\pi/2$ behind the first. The two polarisations are perpendicular, i.e.

$$(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j}) = \mathbf{i} \cdot \mathbf{i} - \mathbf{j} \cdot \mathbf{j} = 1 - 1 = 0, \tag{5}$$

(where we use $\mathbf{i} \cdot \mathbf{j} = 0$). Two perpendicular linearly-polarised waves of equal amplitude and a $\pi/2$ phase difference give circularly polarised light.

The squared amplitude of $E_0 \cos(kz - \omega t)(\mathbf{i} + \mathbf{j})$ is

$$E_0^2 \cos^2(kz - \omega t)(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = E_0^2 \cos^2(kz - \omega t)(\mathbf{i} \cdot \mathbf{i} + \mathbf{j} \cdot \mathbf{j}) = 2E_0^2 \cos^2(kz - \omega t). \tag{6}$$

and similarly of $E_0 \sin(kz - \omega t)(\mathbf{i} - \mathbf{j})$

$$E_0^2 \sin^2(kz - \omega t)(\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j}) = 2E_0^2 \sin^2(kz - \omega t). \tag{7}$$

Hence, the total amplitude is

$$\sqrt{2E_0^2(\cos^2(kz - \omega t) + \sin^2(kz - \omega t))} = \sqrt{2} E_0, \tag{8}$$

and maximum and minimum are the same, confirming the circular nature of the polarisation.

At $z = 0$ and $t = 0$ the wave is $E_0(\mathbf{i} + \mathbf{j})$, so is aligned halfway between the x and y axes. At $z = 0$ and $\omega t = \pi/2$, i.e. 1/4 of a period later the wave is $E_0(-\mathbf{i}+\mathbf{j})$, so is aligned halfway between the y axis and negative x axis. Hence, it is rotating in an anti-clockwise direction.

3. Using Snell's law

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} = \frac{\cos\theta_i - (n_t/n_i)\cos\theta_t}{\cos\theta_i + (n_t/n_i)\cos\theta_t} = \frac{\cos\theta_i - (\sin\theta_i/\sin\theta_t)\cos\theta_t}{\cos\theta_i + (\sin\theta_i/\sin\theta_t)\cos\theta_t}. \tag{9}$$

Multiplying top and bottom by $\sin\theta_t$,

$$r_\perp = \frac{\sin\theta_t \cos\theta_i - \sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t}, \tag{10}$$

which using the standard double angle formula $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$ gives the desired result.

4. For light incident from air $\theta_p = \tan^{-1} 1.4 = 54.5^0$. From Snell's law the angle of transmission is $1.4 \sin\theta_t = \sin 54.5^0 \rightarrow \theta_t = 35.5^0$. This gives

$$r_\perp = \frac{0.58 - 1.4 \times 0.81}{0.58 + 1.4 \times 0.81} = -0.32. \tag{11}$$

For light incident from glass $\theta_p = \tan^{-1}(1/1.4) = 35.5^0$ From Snell's law the angle of transmission is $\sin\theta_t = 1.4 \sin 35.5^0 \rightarrow \theta_t = 54.5^0$, and

$$r_\perp = \frac{1.4 \times 0.81 - 0.58}{1.4 \times 0.81 + 0.58} = 0.32. \tag{12}$$

The angle for total internal reflection is $\sin\theta_c = (n_t/n_i)$ which gives $\sin^{-1}(1/1.4) = 45.6^0$, which is larger than θ_p as we would expect.

5. The wave reflected from the back surface of the film travels $2 \times 1.648 \times 10^{-6} = 3.296 \times 10^{-6}$m. However, this travel is through the medium with $n = 1.555$ so the optical path length is $1.555 \times 3.296 \times 10^{-6} = 5.125 \times 10^{-6}$m. Since the wavelength is $\lambda_0 = 487.99$nm this corresponds to $(5.125 \times 10^{-6})/(487.99 \times 10^{-9}) = 10.5$ wavelengths. Each wavelength corresponds to a phase difference of 2π, so the phase difference between the two reflected waves from this extra path is $\Delta\phi = 10.5 \times 2\pi = 21\pi$.

There is an additional contribution of $\Delta\phi = \pi$ from reflection at the front surface since in this case $n_t > n_i$. Hence, the total phase difference is 22π. This is $11 \times 2\pi$, i.e. is an integral factor of 2π so the two contributions are in phase and add constructively.

PHAS1224 Waves, Optics and Acoustics

Problem Class IV

1. Light with wavelength 600nm passes through two very narrow slits and the interference pattern is observed on a screen 3m away. The second order $m = 2$ bright fringe is at 4.8mm from the centre of the central bright fringe. For what wavelength of light would the third order $m = 3$ bright fringe be seen at the same place?

2. A diffraction grating generates an intensity pattern

$$I = I_0 \frac{\sin^2((\pi/\lambda)Nd\sin\theta)}{\sin^2((\pi/\lambda)d\sin\theta)},$$

where N is the number of slits and d their separation. Explain the position and intensity of the primary maxima. A grating has 50,000 slits over a total width of 75mm. How many primary maxima of yellow light with $\lambda = 700$nm and blue light with $\lambda = 400$nm can be observed respectively?

3. Determine the maximum distance at which a human eye, with a lens diameter of 4mm, is capable of resolving the headlights of a car if the headlights are 1.2m apart when the wavelength of the light is taken to be 500nm.

4. For a spherical concave mirror the distance from the mirror of an object p, the image formed q, and the focal length of the mirror are related by

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

An object placed 300cm from such a mirror generates an image at 150cm. To where must we move the object to obtain the object and image at the same distance?

5. A person has a distance between the lens in their eye and their retina of 2cm. The maximum focusing occurs when the focal length of their lens is 1.98cm. What is their near point (measured from the lens of their eye)? What type of lens in their glasses would they need to obtain a near point at 25cm? Assuming the distance between the lens in their glasses and that in their eye is very small find the focal length of glasses needed to achieve this.

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Problem Class IV

1. The bright fringes are seen when $d\sin\theta = m\lambda$. At a given point on the screen $\sin\theta$ is a constant so if we see the second fringe for light of wavelength λ_1 at the same place as the third fringe for light of wavelength λ_2 then

$$2\lambda_1/d = 3\lambda_2/d \to \lambda_2 = 2/3\lambda_1. \tag{1}$$

So if $\lambda_1 = 600$nm then $\lambda_2 = 400$nm.

2. Primary maximum at $d\sin\theta = m\lambda$ since both $\sin^2((\pi/\lambda)Nd\sin\theta)$ and $\sin^2((\pi/\lambda)d\sin\theta) \to 0$ but ratio

$$\frac{\sin^2((\pi/\lambda)Nd\sin\theta)}{\sin^2((\pi/\lambda)d\sin\theta)} \to \left(\frac{((\pi/\lambda)Nd\sin\theta)}{((\pi/\lambda)d\sin\theta)}\right)^2 \to N^2, \tag{2}$$

and hence intensity $\to I_0 N^2$.

For this grating $d = 75\text{mm}/50,000 = 1.5\times10^{-6}$m. For the primary maxima $d\sin\theta = m\lambda$, so if $\lambda = 700$nm

$$\sin\theta = 0.466m, \tag{3}$$

so there are real solutions for θ for $m = 1$ and 2. If $\lambda = 400$nm, then

$$\sin\theta = 0.266m, \tag{4}$$

so this time there are solutions for θ if $m = 1$, 2 or 3.

3. For a circular aperture of diameter D the corresponding minimum angular separation is $1.22\lambda/D$, and at a distance L the angular separation of two points a distance d apart is given by $\tan\theta = d/L$ which becomes $\theta = s/L$ if $\theta \ll 1$. Therefore, to resolve the headlights we must have

$$d/L > 1.22\lambda/D \to L < dD/(1.22\lambda). \tag{5}$$

Inputing the values for this situation

$$L < \frac{1.2\times4\times10^{-3}}{1.22\times5\times10^{-7}} = 8000\text{m}. \tag{6}$$

4. Simply making the substitution we obtain

$$\frac{1}{f} = \frac{1}{300\text{cm}} + \frac{1}{150\text{cm}} = \frac{450}{45,000\text{cm}} \to f = 100\text{cm}. \tag{7}$$

If we the insist that $p = q$ then

$$\frac{1}{p} + \frac{1}{p} = \frac{2}{p} = \frac{1}{f}, \tag{8}$$

and so $p = 2f = 200\text{cm}$.

5. The retina distance is q in this situation, and the near point is the value of p when there is most focusing, i.e. when $f = 1.98\text{cm}$. Hence,

$$\frac{1}{p} = \frac{1}{f} - \frac{1}{q} = \frac{1}{1.98\text{cm}} + \frac{1}{2\text{cm}} = \frac{1}{198\text{cm}}, \tag{9}$$

and the near point $p = 198\text{cm}$.

We want the near point of the lens plus the glasses to be 25cm, i.e. if $q = 2\text{cm}$, then $p = 25\text{cm}$. This requires a combined focal length of f_{com} given by

$$\frac{1}{f_{\text{com}}} = \frac{1}{25\text{cm}} + \frac{1}{2\text{cm}} = \frac{27}{50\text{cm}}. \tag{10}$$

So the combined focal length is $f_{\text{com}} = 50/27 = 1.85\text{cm}$. The image from the first lens (the glasses) forms the object for the second. However, if the distance between the two lenses is negligible the image distance for the first lens is the same as the object distance for the second but with a minus sign since the object for the second lens is behind the lens, i.e. we have $p_2 = -q_1$. Using the lens equation

$$\frac{1}{q_1} = \frac{1}{f_1} - \frac{1}{p_1} \tag{11}$$

and also

$$\frac{1}{q_2} = \frac{1}{f_2} - \frac{1}{p_2} = \frac{1}{f_2} + \frac{1}{q_1} = \frac{1}{f_2} + \frac{1}{f_1} - \frac{1}{p_1}. \tag{12}$$

We can rearrange this to show that

$$\frac{1}{q_2} + \frac{1}{p_1} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{f_1 + f_2}{f_1 f_2} \equiv \frac{1}{f_{\text{com}}}. \tag{13}$$

The combined focal length of two lenses with negligible separation is therefore $f_{\text{com}} = f_1 f_2/(f_1 + f_2)$.

In this case we want $f_{\text{com}} = 1.85\text{cm}$ and have $f_2 = 1.98$ cm so

$$\frac{1}{f_1} = \frac{1}{1.85\text{cm}} - \frac{1}{1.98\text{cm}} \rightarrow f_1 = 28.2\text{cm}. \tag{14}$$

Hence, the glasses are converging with focal length 28.2cm.