UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

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ASSESSMENT PATTERN : **PHAS3201A**

MODULE NAME : **Electromagnetic Theory**

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TIME : **10:00**

TIME ALLOWED : **2 Hours 30 Minutes**

2009/10-PHAS3201A-001-EXAM-95

TURN OVER

Answer ALL SIX questions in Section A and TWO questions from Section B

The numbers in square brackets at the right-hand side of the text indicate the provisional allocation of maximum marks per question or sub-section of a question.

You may find the following constants and theorems useful.

$$\begin{aligned} \epsilon_0 &= 8.85 \times 10^{-12}\mathrm{F/m} \\ m_e &= 9.11 \times 10^{-31}\mathrm{kg} \\ e &= 1.60 \times 10^{-19}\mathrm{C} \\ c &= 3.00 \times 10^{8}\mathrm{m/s} \end{aligned}$$

For any vector $\mathbf{F}$, $\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

In cylindrical polar coordinates, $\nabla \times \mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \frac{1}{R} \begin{vmatrix} \mathbf{i}_R & R\mathbf{i}_\phi & \mathbf{i}_z \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_R & RF_\phi & F_z \end{vmatrix}$$

For any vector function which can be written $\mathbf{C}(\mathbf{r}, t) = \mathbf{D} \exp i\,(\mathbf{k} \cdot \mathbf{r} - \omega t)$ where $\mathbf{D}$ is a constant, then:

$$\begin{aligned} \nabla \cdot \mathbf{C} &= i\mathbf{k} \cdot \mathbf{C} \\ \nabla \times \mathbf{C} &= i\mathbf{k} \times \mathbf{C} \\ \nabla^2 \mathbf{C} &= -k^2 \mathbf{C} \end{aligned}$$

Useful four-vectors and operators:

$$\begin{aligned} \partial_\mu &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t}\right) \\ x^\mu &= (x, y, z, ct) \\ a^\mu &= (A_x, A_y, A_z, \frac{\phi}{c}) \\ j^\mu &= (J_x, J_y, J_z, c\rho) \\ \text{D'Alembertian } \Box &= \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \end{aligned}$$

The Lorentz transform from a reference frame S to a reference frame S$'$ with relative velocity $(v, 0, 0)$ for a general contravariant four-vector $f^\mu = (f^1, f^2, f^3, f^4)$ can be written as:

$$\begin{aligned} f'^1 &= \gamma\,(f^1 - \beta f^4) \\ f'^2 &= f^2 \\ f'^3 &= f^3 \\ f'^4 &= \gamma\,(f^4 - \beta f^1)\,, \end{aligned}$$

where $\beta = v/c, \gamma = 1/\sqrt{1-\beta^2}$.

SECTION A

1. (a) Describe at the atomic level how magnetisation arises in a linear magnetic material in an applied magnetic field. [4]

   (b) Explain *briefly* the difference between paramagnetism and diamagnetism at the atomic level. [2]

2. (a) Give an equation defining the polarisation, P, in terms of the electric field E and other quantities. Define any symbols you use. [2]

   (b) Hence show how, in a linear material, the electric displacement D can be written in terms of the electric field as $\mathbf{D} = \epsilon\mathbf{E}$. Define ϵ. [4]

3. For a linear material with conductivity g, permittivity ϵ and permeability μ, derive the wave equation for the electric field of an electromagnetic wave:

$$\nabla^2\mathbf{E} - g\mu\frac{\partial\mathbf{E}}{\partial t} - \epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

   from Faraday's law and the Ampere-Maxwell equation. [6]

4. (a) What is the meaning of skin depth for a conducting material ? [2]

   (b) Show that, for a good conductor, the skin depth can be written as:

$$\delta = \sqrt{\frac{2}{\mu g\omega}}$$

   and specify what condition must apply for a material to be considered a good conductor. (You may find it helpful to consider a plane wave and use the wave equation above). [5]

5. (a) Define the surface and bulk magnetisation current densities in terms of the magnetisation M in a magnetic material. [4]

   (b) Show how, in the absence of electric fields, Maxwell's equation $\nabla \times \mathbf{B} = \mu_0\mathbf{J}$ can be re-written in terms of H and the free current density $\mathbf{J}_{\text{free}}$. [4]

6. (a) Derive the boundary conditions on E and H at the interface between two materials in the absence of free surface currents. [4]

   (b) How will the boundary conditions be altered in the presence of free surface currents ? [3]

SECTION B

7. (a) Using the integral form of Ampère's law ($\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$), show that the magnetic field in the central region inside a long solenoid lying along the z-axis with n turns per unit length carrying a current I can be written: [4]

$$\mathbf{B} = \mu_0 n I \mathbf{i}_z$$

(b) What is the magnetic field inside a cylindrical ferromagnet (with the same dimensions and orientation as the solenoid) with magnetization $\mathbf{M} = M\mathbf{i}_z$? In what ways do the fields and currents for a solenoid and for a cylindrical ferromagnet resemble each other ? [4]

(c) The vector potential of a small loop of wire, radius a, centred on the origin and lying in the x-y plane, and which carries current I, can be written in cylindrical polar coordinates:

$$A_\phi = \frac{\mu_0 I a^2}{4} \frac{R}{(a^2+z^2)^{3/2}} (A_z = 0, A_R = 0)$$

Show that the magnetic field from this current loop at $\mathbf{r} = (R, \phi, z)$ is: [4]

$$\begin{aligned} B_R &= \frac{\mu_0 I a^2}{4} \frac{-3zR}{(a^2+z^2)^{\frac{5}{2}}} \\ B_z &= \frac{\mu_0 I a^2}{4} \frac{2}{(a^2+z^2)^{\frac{3}{2}}} \\ B_\phi &= 0 \end{aligned}$$

(d) The force on a magnetic dipole (with moment $\mathbf{m}$) in a magnetic field $\mathbf{B}$ can be written:

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

Explain why there will be no net force on a magnetic dipole placed in the central region of a long solenoid no matter what its orientation, and describe briefly what will happen to dipoles oriented along x or y. [4]

(e) A solenoid of radius a and length L lies along the z axis from $z = 0$ to $z = L$ with N turns in total , but with a *varying* number of turns per unit length, given by:

$$n = 2Nz/L$$

i. Using the expression for the magnetic field of a current loop above, show that the z-component of the magnetic field at a point z can be written:

$$B_z(z) = \frac{\mu_0 I a^2}{4} \frac{N}{L} \int_0^L \frac{2z'}{(a^2 + (z'-z)^2)^{3/2}} dz'$$

and evaluate the integral (you may find $x = a\tan\theta$, $dx = a\sec^2\theta d\theta$, $1 + \tan^2\theta = \sec^2\theta$ and $\sin\theta = \tan\theta/\sqrt{1+\tan^2\theta}$ useful). [10]

ii. Hence show that there will be a force acting on a magnetic dipole with moment $\mathbf{m} = (0, 0, m)$ at position z inside the solenoid. [2]

iii. Find an expression for B_z far from the solenoid. [2]

8. (a) The Lorenz gauge condition is written:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

Show how the left-hand side can be written as the scalar product of two four vectors. Comment on the significance of this, and contrast with the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. [6]

(b) Starting from the Maxwell equations with sources in free space ($\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0\mu_0 \partial \mathbf{E}/\partial t$) and working in the Lorenz gauge, derive the wave equations for the potentials:

$$\begin{aligned} \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} \\ \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -\rho/\epsilon_0 \end{aligned}$$

[8]

(c) Show how the wave equations just derived can be written in Lorentz invariant form as the d'Alembertian operating on a four vector, and explain why the d'Alembertian is a Lorentz-invariant operator. [4]

(d) The wavevector $\mathbf{k}$ and angular frequency ω form a *covariant* four-vector, $K_\mu = (k_x, k_y, k_z, \omega/c)$. Explain what this implies about the phase of a plane wave under Lorentz transformation. [4]

(e) The Lorentz transforms for the electric and magnetic fields from a frame S to another frame S' moving with velocity $\mathbf{v}$ relative to S are given by:

$$\begin{aligned} E'_{\parallel} &= E_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma \mathbf{E}_{\perp} + \gamma \mathbf{v} \times \mathbf{B} \\ B'_{\parallel} &= B_{\parallel} \\ \mathbf{B}'_{\perp} &= \gamma \mathbf{B}_{\perp} - \frac{1}{c^2}\gamma \mathbf{v} \times \mathbf{E}, \end{aligned}$$

where the subscripts $\parallel$ and $\perp$ refer to components of the field parallel and perpendicular to the velocity.

Consider a plane electromagnetic wave propagating in the z direction in a frame S with wavevector $\mathbf{k} = (0, 0, k)$, angular frequency ω and electric and magnetic fields:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{i}_x \\ \mathbf{B}(\mathbf{r}, t) &= B_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathbf{i}_y \end{aligned}$$

i. Transform $\mathbf{k}, \mathbf{E}$ and $\mathbf{B}$ to a frame S$'$ moving with velocity $\mathbf{v}' = (0, 0, v)$ and comment on the relative orientations of $\mathbf{k}, \mathbf{E}$ and $\mathbf{B}$ for each of the frames. [4]

ii. Now transform $\mathbf{k}, \mathbf{E}$ and $\mathbf{B}$ to a frame S$''$ moving with velocity $\mathbf{v}'' = (v, 0, 0)$ and comment on the relative orientations of $\mathbf{k}, \mathbf{E}$ and $\mathbf{B}$ [4]

9. (a) Describe the key characteristics of a plasma. [2]

(b) The plasma frequency is defined as:

$$\omega_p = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}$$

Define the symbols N_e, ϵ_0, m_e and e. [2]

(c) Use the continuity equation ($\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$) and the polarisation charge density to show that a time-varying polarisation is equivalent to a current density. [2]

(d) An electromagnetic plane wave of frequency ω passing through a plasma will induce a polarisation current density:

$$\mathbf{J}_P = i\left(\frac{N_e e^2}{m_e \omega}\right)\mathbf{E}$$

By considering the time derivative of the displacement field and using the definition of $\mathbf{D}$ in terms of $\mathbf{E}$ and the polarisation $\mathbf{P}$, show that the relative permittivity of a plasma can be written:

$$\epsilon_r = 1 - \frac{\omega_P^2}{\omega^2}$$

[8]

(e) What happens to the electric field amplitude as $\omega \rightarrow \omega_P$? What will happen physically in the plasma ? [2]

(f) Show that the dispersion relation is $k^2 = \omega^2(1 - \omega_P^2/\omega^2)/c^2$. [2]

(g) Explain, using appropriate equations, what will happen when an electromagnetic plane wave propagates towards a plasma when:

i. $\omega < \omega_P$ [4]

ii. $\omega > \omega_P$ [4]

(h) If a plasma was to be used as a shield against a laser in the visible spectrum (400-700nm wavelength light), what is the minimum electron density that should be used ? What thickness of this shield would be needed to reduce the power of a 10 kW beam with wavelength 500nm to below 1 W ? [4]

10. (a) For a plane wave in vacuum with electric field $\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$, use one of Maxwell's equations to show that the magnetic field can be written:

$$\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega}$$

[4]

(b) Using this result and another equation, or otherwise, show that k, the magnitude of the wavevector, and ω are related by $\omega/k = c$ [2]

(c) Now consider the same plane wave in a linear dielectric with relative permittivity ϵ_r and relative permeability $\mu_r = 1$. Show that the wavevector and angular frequency are now related by $k = n\omega/c$ and give an expression for n. [4]

(d) Using the first parts of the question, show that the magnitude of the Poynting vector is given by: [4]

$$N = \sqrt{\frac{\epsilon_0}{\mu_0}} n E_0^2$$

(e) Now consider the planewave incident on an interface between vacuum and the surface of the dielectic. The Fresnel relations can be written:

$$\begin{aligned} r_\parallel &= \frac{n'\cos\alpha - n\cos\alpha'}{n'\cos\alpha + n\cos\alpha'} \\ r_\perp &= \frac{n\cos\alpha - n'\cos\alpha'}{n\cos\alpha + n'\cos\alpha'} \\ t_\parallel &= \frac{2n\cos\alpha}{n'\cos\alpha + n\cos\alpha'} \\ t_\perp &= \frac{2n\cos\alpha}{n\cos\alpha + n'\cos\alpha'} \end{aligned}$$

Draw a diagram of the interface and define the symbols $n, n', \alpha, \alpha', r_\parallel, r_\perp, t_\parallel, t_\perp$ [6]

(f) The reflected and transmitted *intensity coefficients* R and T are given by the ratios of the magnitudes of the Poynting vectors *normal* to the interface. For the case of the electric field lying in the plane of the wave vectors,

i. Show that $R_\parallel = r_\parallel^2$ [2]

ii. Show that $T_\parallel = \frac{n'\cos\alpha'}{n\cos\alpha} t_\parallel^2$ [4]

iii. Find $R + T$ [4]