UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics M254: Problem-solving in Pure Mathematics

COURSE CODE	:	**MATHM254**
UNIT VALUE	:	**0.50**
DATE	:	**28-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

04-C0966-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let a_1, b_1, c_1 and a_2, b_2, c_2 be non-zero real numbers such that each of
$$(a_1x + b_1)^2 + (a_2x + b_2)^2 \quad \text{and} \quad (b_1x + c_1)^2 + (b_2x + c_2)^2$$
is the square of a polynomial of degree one. Show that $(c_1x + a_1)^2 + (c_2x + a_2)^2$ is also the square of a polynomial of degree one.

2. Let $f : \mathbb{R} \to (0, \infty)$ be a monotone increasing function. Show that there is an $x \in \mathbb{R}$ such that
$$f\left(x + \frac{1}{f(x)}\right) < 2f(x)$$

3. A disc is placed on each square of an $n \times n$ chess-board. Each disc has a red and a blue face. Two players play the following game. They move alternately. The one who moves picks out a blue disc and turns over every disc in the rectangle whose upper left corner is the square containing the selected disc and whose lower right corner is the lower right corner of the whole board. The first player who cannot find a blue disc to turn over loses the game. Prove that the game always ends in finitely many steps. Determine those initial positions for which the first player wins and show that the result of the game depends only on the initial position and not on how the players play.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function. Show that the set of points x at which f is not differentiable but the limits
$$\lim_{y \to x-} \frac{f(y) - f(x)}{y - x} \quad \text{and} \quad \lim_{y \to x+} \frac{f(y) - f(x)}{y - x}$$
both exist is countable.

5. A set T of positive rational numbers is called complete if for each $\frac{p}{q} \in T$, where p, q are positive integers, both $\frac{p}{p+q} \in T$ and $\frac{q}{p+q} \in T$. Find all positive rational r such that every complete set containing r contains all rational numbers strictly between 0 and 1.

END OF PAPER