UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATHGM02**

ASSESSMENT PATTERN : **MATHGM02A**

MODULE NAME : **Nonlinear Systems**

DATE : **01-Jun-09**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

*All questions may be attempted but **only** marks obtained on the best **four** solutions will count.*

*The use of electronic calculators is **not** permitted in this examination.*

1. a) The equation of motion for the roll of a ship is determined as

$$M\ddot{\theta} + B\dot{\theta} + Mg\theta(\theta_v - \theta)(\theta_v + \theta) = 0$$

where θ is the roll angle, θ_v is the angle of vanishing stability (the angle beyond which the ship no longer returns to the upright position), g the usual gravitational constant, M the mass, B a term that represents damping and the overdot represents differentiation with respect to time.

By suitable scaling write this equation in the form $x'' + \bar{b}x' + x - x^3 = 0$ where the dash now denotes differentiation with respect to your scaled time.

b) Using geometrical arguments determine the steady state equilibrium solutions, assess their stability and sketch the basin of attraction for any stable equilibrium solutions of the system given by

$$\ddot{x} + b\dot{x} + x - x^3 = 0$$

where the constant term b is positive but much less than critical damping. You should add comments and justify your conclusions.

c) As the parameters F and ω are both varied, discuss the bifurcations that might occur in the driven system

$$\ddot{x} + b\dot{x} + f(x) = F\sin(\omega t)$$

where $f(x)$ is a nonlinear function.

c) Discuss EITHER of the two terms below

i) fractal basin boundary OR ii) global bifurcation

PLEASE TURN OVER

2. a) Write down the normal form equation for a one dimensional ordinary differential equation undergoing a fold bifurcation and demonstrate how it gets its name by indicating the paths of equilibrium solutions as a parameter is varied.

b) The dynamics of a system is governed by the equation

$$\dot{x} = (x-1)^2(x-2)^2 - \mu$$

where μ is a real parameter.

As μ is varied, determine the number and position of all possible equilibrium solutions and, using *graphical* considerations, establish their stability.

Hence sketch a plot of the path of these equilibria as μ is varied and determine if any bifurcations take place.

c) A second system is governed by the complex equation

$$\dot{z} = z\left\{1 + 2i - \mu|z| + |z|^2\right\}$$

where μ is a positive, real parameter.

Find the values of μ for which periodic solutions exist and hence determine any bifurcations that take place.

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3. a) A *general* continuous dynamical system is governed by the equation

$$\dot{\underline{x}} = \underline{F}(\underline{x})$$

State the conditions for an equilibrium solution to exist and also define what we mean when we say that an equilibrium solution is locally asymptotically stable.

b) A *linear* continuous dynamical system is governed by the equation

$$\dot{\underline{x}} = A\underline{x}$$

where A is an $n \times n$ matrix and the only equilibrium solution is at the origin.

State a condition for the origin to be asymptotically stable.

c) A *linear 2-dimensional* continuous dynamical system is governed by the equation

$$\dot{\underline{x}} = J\underline{x}$$

where J is now a 2×2 matrix.

Briefly discuss generic bifurcations as a single parameter is varied.

d) Determine the equilibrium solutions for the system governed by the equations

$$\dot{x} = \mu x + y - xy$$

$$\dot{y} = x - y$$

where μ is a real parameter.

By comparison with the normal form equations of standard bifurcations, or otherwise, establish the bifurcation that takes place as μ is varied.

When $\mu = 1$ determine the local dynamics close to any equilibrium solutions and hence sketch a global phase portrait.

PLEASE TURN OVER

4. a) For a general iterative one-dimensional map given by

$$x_{n+1} = \phi(x_n)$$

i) Explain how you would find any fixed points and determine their stability,

ii) Briefly explain how you would establish stability of a fixed point x_F was such that

$$|\phi'(x_F)| = 1$$

iii) Discuss how you would find a 2-cycle, or period-2 orbit, and state how you would determine the stability of the 2-cycle.

b) For the iterative map given by

$$x_{n+1} = x_n^2 - 1$$

find all the fixed points and any 2-cycles of the map and determine their stability.

c) When considering the change in gene frequency (x_n) from one generation of locusts to the next generation a proposed genotype selection model is given by

$$x_{n+1} = \frac{x_n f(x_n)}{x_n f(x_n) + 1 - x_n},$$

where the frequency x is always positive. The function f is such that $f(x) = e^{\beta(1-2x)}$, where β is a positive parameter.

i) For this model, show that there are three possible fixed points and determine their stability as the parameter β is varied.

ii) Classify any bifurcations of the fixed points that occur and plot a likely bifurcation diagram (you do ***not*** have to analytically find the path of any 2-cycles).

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5. a) When considering solutions $u(x,t)$ of general *linear* homogeneous partial differential equations we often assume a harmonic wave in the form

$$u = \mathrm{Re}\left\{Ae^{i(kx-\omega t)}\right\},$$

and establish a so-called dispersion relationship which gives the angular frequency, ω, as a function of k, where k is the wave number (the spatial analogue of frequency).

Given the following relationships, in each case state the effect on the subsequent solutions, paying particular attention to dispersion and stability:-

i) $\omega = 2k$ ii) $\omega = k + k^3$ iii) $\omega = ak + i\left(k^2 - k^4\right)$

b) Consider travelling-wave solutions of the *nonlinear* equation

$$u_{tt} - u_{xx} + u^3 - u = 0$$

i) By letting $u = f(\xi)$ and $\xi = x - ct$, where c is the constant phase velocity, show that the travelling-wave solutions satisfy

$$\left(c^2 - 1\right)\left(f'\right)^2 = f^2 - \frac{1}{2}f^4 + D$$

where D is an arbitrary constant.

ii) Rewrite the ordinary differential equation in (i) in the form

$$\left(f'\right)^2 = F(f)$$

and show that $F(f)$ has two double roots when $D = -\frac{1}{2}$.

iii) Hence, or otherwise, show that when f is 'trapped' between the two double roots of $F(f)$ we can find bounded solutions for f.

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