UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATHGM02**
ASSESSMENT PATTERN	:	**MATHGM02A**
MODULE NAME	:	**Nonlinear Systems**
DATE	:	**29-Apr-08**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2007/08-MATHGM02A-001-EXAM-21

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*All questions may be attempted but **only** marks obtained on the best **four** solutions will count.*

*The use of electronic calculators is **not** permitted in this examination.*

1. a) A physical apparatus consists of a ball rolling on a tracked surface. The motion of the ball is governed by a scaled equation of the form

$$\ddot{x} + b\dot{x} + x - x^2 = 0.$$

Using geometrical arguments determine the steady state equilibrium solutions, assess their stability and sketch the basin of attraction for any stable solutions.

b) Discuss the role of the term b in the above equation and how you might estimate its value.

c) The apparatus us now placed on an experimental shaking table so that the subsequent motion is governed by the forced equation

$$\ddot{x} + b\dot{x} + x - x^2 = F\sin(\omega t)$$

Where F and ω are constants with $\omega \neq 0$.

Discuss how this forcing affects the system response as both F and ω are varied.

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2. a) When considering a nonlinear system controlled by a parameter state what we mean by the term bifurcation and briefly discuss how we might determine whether a local or global bifurcation was about to take place.

b) Consider the 2-dimensional system of equations given by

$$\dot{x} = \mu x + y - 3xy$$

$$\dot{y} = x - y,$$

where μ is a real parameter.

By considering the type and stability of steady state equilbrium solutions that result as μ is varied, determine the type of bifurcation that takes place.

c) i) State what we mean when we say a bifurcation is structurally stable.

ii) The system defined by

$$\dot{x} = \mu x - x^2$$

is the normal form for a flow undergoing a transcritical bifurcation.

Demonstrate whether this bifurcation is structurally stable or not.

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3. a) A dynamical system is governed by the two equations

$$\dot{x} = x(2 - x - y)$$

$$\dot{y} = y(4 - 3x - y),$$

where the variables x and y are always taken to be positive.

(i) State the conditions for an equilibrium solution to exist and *analytically* find all four equilibria of this system.

(ii) By considering the eigenvalues of the Jacobian matrix of partial derivatives for the system, determine the stability of the equilibrium solutions found in part (i). For any saddle solutions, by further considering the eigenvectors of the Jacobian matrix, visualise the nature of the dynamics local to any equilibrium states and hence give a sketch of the global phase portrait in the (x, y) phase space.

(iii) Define what we mean by a basin of attraction and, using the phase space diagram obtained in part (ii), denote the basins of attraction for any stable solutions found. Explain how you might find the curves that define a basin of attraction.

b) The Lorenz system of equations is given by

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz,$$

where the parameters σ, r and b are usually positive.

Show that the function

$$V = \frac{1}{\sigma}x^2 + y^2 + z^2$$

forms a Lyapunov function and hence explain the significance for the existence of periodic solutions for $r < 1$.

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4. b) The Hénon map is given by

$$x_{n+1} = 1 + y_n - ax_n^2$$
$$y_{n+1} = bx_n,$$

where a and b are constants.

i) State how you would find any fixed points of this map and find any fixed points when $a = 0.08$ and $b = 0.3$.

ii) State how you would determine the stability of any fixed points of a general 2-dimensional map and sketch a region in the plane of the Determinant and the Trace of the Jacobian matrix, which guarantees stability. Hence determine whether the fixed points found in part (i) above for the Hénon map are stable or not (note that you need *not* explicitly find the eigenvalues).

b) The quadratic map is given by

$$x_{n+1} = \mu - x_n^2$$

Show that period-2 fixed points exist for $\mu > 3/4$.

c) The logistic map is given by

$$x_{n+1} = \mu x_n(1 - x_n).$$

When $\mu = 4$, using a change of variable $x = \frac{1}{2}\{1 - \cos(\theta)\}$ allows us to write successive iterates as

$$x_n = \frac{1}{2}\{1 - \cos(2^n \theta_0)\},$$

given a particular initial condition that determines θ_0.

Using this transformed version of the logistic map find the Lyapunov exponent which measures the rate of exponential divergence of nearby starts and hence determine whether the map is behaving chaotically or not.

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5\. a) The velocity, $u(x,t)$, of a flame front may be modelled the Kuramoto-Sivashinsky partial differential equation given by

$$u_t + \gamma(u u_x + u_{xx}) + 4u_{xxxx} = 0.$$

Linearise about $u = 0$ to give the steady state solution satisfying $u(0,t) = 0$ as

$$u(x,t) = \frac{4c_3}{\gamma} - \frac{4}{\gamma}\left\{c_3 \cos\left(\frac{x\sqrt{\gamma}}{2}\right) + c_4 \sin\left(\frac{x\sqrt{\gamma}}{2}\right)\right\},$$

where c_3 and c_4 are constants.

b) By imposing a periodic boundary condition that $u(x,t) = u(x+2\pi,t)$ show that the primary bifurcations of the nonlinear problem occur when $\gamma = 4j^2$, where $j = 1,2,3....$.

c) When considering travelling-wave solutions $u(x,t)$ of the partial differential equations we assume a harmonic wave in the form

$$u = \mathrm{Re}\left\{Ae^{i(kx-\omega t)}\right\},$$

and establish a so-called dispersion relationship which gives ω as a function of k. Discuss the following relationships and their effect on the subsequent solutions, stating the phase velocity in each case :-

i) $\omega = k$, ii) $\omega = -k^3$, iii) $\omega = k - ik^2$.

d) i) Define what we mean by a soliton.

ii) When considering a travelling wave solution $u(x,t) = f(x-t)$ of the equation

$$u_t - u_{xxx} + 3u_x u^2 = 0,$$

show that this leads to a relationship in the form

$$\frac{(f')^2}{2} = \frac{f^4}{4} - \frac{f^2}{2} - Af - B.$$

iii) By choosing the constants as $A = 0$ and $B = -1/4$, show that this relation simplifies so that f satisfies

$$f' = \pm\frac{\sqrt{2}}{2}\left(f^2 - 1\right).$$

iv) Hence, show that $f = \mp\tanh\left\{\frac{(x-t)\sqrt{2}}{2}\right\}$ is a solution.

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