UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATHGM02**

MODULE NAME : **Nonlinear Systems**

DATE : **03-May-07**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATHGM02A-001-EXAM-21

TURN OVER

***All** questions may be attempted but **only** marks obtained on the best **four** solutions will count.*

*The use of electronic calculators is **not** permitted in this examination.*

1. a) The roll motions of a model ship in still water are to be modelled using the equations
$$m\ddot{\theta} + c\dot{\theta} + f(\theta) = 0,$$
where θ is a measure of the angle of roll, m is the mass of the vessel, c a roll damping coefficient and $f(\theta)$ a restoring moment given by
$$f(\theta) = 2\theta - 2\theta^3.$$
Plot the restoring moment and the governing potential energy of the system. Hence plot a sketch of the phase portrait when the damping coefficient is zero, by plotting the path of some transient trajectories including some key trajectories that configure the space.

b) Define what we mean by a basin of attraction and, for a small, positive value c, plot the basin of attraction for the stable equilibrium found in part (a) by plotting the phase space as before but now with $c \neq 0$.

c) A test in a wave tank reveals that the ship is not symmetrically balanced and a small positive moment of 2ε is required to achieve a zero roll angle. That is, in reality
$$f(\theta) = 2\theta - 2\theta^3 + 2\varepsilon.$$
Justify any changes that the addition of this small term makes to the fixed, equilibrium points and the governing potential energy function and hence repeat the procedures above for this new system to plot the restoring moment, the governing potential energy. Sketch the phase portrait when the damping coefficient is both zero and non-zero, but still small.

d) For the non-symmetric system, sketch the phase portrait to show how it might alter as the damping coefficient is increased from zero.

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2. a) When used in conjunction with dynamical systems, explain the term autonomous. State the requirements for an autonomous one-dimensional flow to have a stable equilibrium point. Explain the term bifurcation.

b) Find any equilibrium points of the following system and discuss their stability

$$\frac{dx}{dt} = x^2 - 4x - 5.$$

c) Consider one dimensional flows for the evolution of the real variable x

$$\frac{dx}{dt} = f(x,\mu).$$

Write down the normal form equation for a typical system undergoing a transcritical bifurcation and verify that an exchange of stability takes place as a parameter, μ say, is varied.

d) Explain the term structurally stable bifurcation and by perturbing both the variable x and μ, show that the saddle-node bifurcation is structurally stable.

e) A physical apparatus, designed to be close to a single degree of freedom system, is performing continuous oscillations due to a periodic driving force. The system may be disturbed and the subsequent transient monitored by a data logger that stroboscopically samples the response at times corresponding to multiples of the driving period to give a set of data points of position and velocity.

State how these sets of discrete data may be used to determine whether a bifurcation is about to occur. State what types of bifurcation are possible

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3. a) A flow in two dimensions is given by the linear system of equations

$$\dot{x} = ax + by \qquad \dot{y} = cx + dy.$$

State under what conditions a unique fixed, equilibrium point exists, and state what the equilibrium solution is under such conditions.

b) By considering the eigenvalues of the 2x2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant (D) and trace (T), determine for what values of D and T the local behaviour of the flow near the origin is a saddle, a node or a focus, indicating the stability.

c) Now consider a nonlinear flow in two dimensions given by the system of equations

$$\dot{x} = f_1(x,y) \qquad \dot{y} = f_2(x,y).$$

State the requirement for an equilibrium to exist and also state how you would determine when such an equilibrium was locally asymptotically stable.

d) Consider the system of equations

$$\dot{x} = x^2 + y^2 - 2a^2 \qquad \dot{y} = x^2 - y^2.$$

For $a \neq 0$, find all the equilibrium points and by consideration of the trace and determinant of the Jacobian matrix alone (i.e. you do not have to explicitly find the eigenvalues and the eigenvectors) determine their local stability.

e) Using your solutions from part (c) sketch the phase portrait of the system indicating the behaviour of transient solutions, including the stable and unstable manifolds of the equilibrium solutions.

f) Discuss the existence of equilibrium states and the evolution of the system if $a = 0$.

PLEASE TURN OVER

4. In the following x is a real variable. Let $\phi(x):\Re \to \Re$ be a continuously differentiable, one-dimensional discrete dynamical system mapping the set of real numbers to itself. That is the map is given by $x_{i+1} = \phi(x_i)$.

a) Write down the condition for an equilibrium solution of $\phi(x)$ to exist and state the sufficient condition for such an equilibrium to be locally asymptotically stable.

b) Investigate the equilibria, and their stability, for the map

$$x_{i+1} = \frac{rx_i}{(1+x_i)^2},$$

with $r > 0$ and avoiding the singularity at $x_i = -1$.

When $r = 1$ you should explain how you might determine the stability, but you do not need to perform such an analysis.

Plot a diagram of the position of any equilibria as the parameter r is varied, also denoting the stability of such solutions, and state the form of the bifurcation that takes place when $r = 1$.

c) Define what we mean by a period-2 solution and show how we find if such a solution exists for a general map $x_{i+1} = \phi(x_i)$.

For the nonlinear map of part (b) show that the path of period-2 solutions is given by

$$x = -1 - r \pm \sqrt{r(1+r)}.$$

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5. a) A modelling process indicates that a media can exhibit waves that are governed by a partial differential equation of the form

$$u_t + u_{xx} + u_{xxxx} = 0,$$

where x and t are real variables for which t varies between 0 and ∞. Assuming a harmonic wave

$$u = \mathrm{Re}\left\{Ae^{i(kx-\omega t)}\right\},$$

with ω complex, derive the dispersion relation and determine the range of real valued wave-numbers (k) for which the waves grow (become unstable).

b) Consider the travelling-wave solutions $u(x,t)$ of the nonlinear wave equation

$$u_t u_x + \frac{1}{2} u u_{xx} + \frac{1}{3} u^3 = 0,$$

which have a constant phase speed c. Using the substitution $\eta = x - ct$, i.e. with $k = 1$, show that the shape function $u = f(\eta)$ satisfies the ordinary differential equation

$$\frac{1}{2} f f'' - c(f') + \frac{1}{3} f^3 = 0.$$

c) Verify that $u = A\left(B + \eta^2\right)^{-1}$ is a solution of the ordinary differential equation in part (b) for real constants A and B and determine any requirements on A, B and c.

d) Discuss the evolution of two or more solutions, with different characteristics, for both of the partial differential equations above.

END OF PAPER