UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

M.Sc. *PG Dip*

Frontiers in Mathematical Modelling and its Applications

COURSE CODE	:	**MATHGM05**
DATE	:	**15-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

06-N0135-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) You are given a sample of a linear viscoelastic material and an oscillatory rheometer. Describe briefly how you would measure the linear rheology of the material.

   (b) Given the definition

$$G^*(\omega) = i\omega \int_0^\infty G(s) \exp\left[-i\omega s\right] \mathrm{d}s,$$

   of the complex shear modulus, give expressions for the storage modulus G' and loss modulus G'' of the material.

   (c) Prove for the experiment you described in (a) that the shear stress is

$$\sigma = G'\gamma(t) + \frac{G''}{\omega} \dot{\gamma}(t)$$

   where γ is the shear displacement and $\dot{\gamma}$ the shear rate.

   (d) Calculate the storage and loss moduli for a material with a single exponential relaxation time, τ. For what value of ω are the two moduli equal?

2. (a) A vortex of strength κ is present at the point z_0 in a region R_z of the complex z-plane so that the complex potential $w(z)$ is analytic everywhere in R_z except at the point z_0, with

$$w(z) \longrightarrow -(\mathrm{i}\kappa/2\pi)\log(z - z_0), \quad \text{as} \quad z \longrightarrow z_0.$$

The associated complex velocity is

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z}.$$

The region R_z is mapped into the region R_ζ of the complex ζ-plane by the conformal mapping, analytic within R_z, given by

$$\zeta = f(z), \qquad z = f^{-1}(\zeta) = F(\zeta).$$

(i) Show that the vortex at z_0 maps to a vortex at $\zeta_0 = f(z_0)$ and find its strength.

(ii) Find an expression for the fluid velocity at a point $\zeta \neq \zeta_0$ in R_ζ in terms of $w(z)$.

(iii) Find an expression for the velocity in R_ζ of the vortex at ζ_0 in terms of $w(z)$, given that there is no self-induced motion of a point vortex.

(b) Suppose that there are point vortices of strength $\kappa_1, \kappa_2, \ldots, \kappa_n$ with instantaneous positions $\zeta_1, \zeta_2, \ldots, \zeta_n$ moving in infinite space and that the vorticity is zero elsewhere. Then the quantity

$$H(\zeta_1, \zeta_2, \ldots, \zeta_n) = -(1/4\pi)\sum_{i=1}^{n}\sum_{j=1,(j\neq i)}^{n} \kappa_i\kappa_j \log|\zeta_j - \zeta_i|$$

is a constant of the motion.

By considering vortices of strength $\pm\kappa_0$ at $\zeta = \zeta_0, \bar{\zeta}_0$, where $\bar{\ }$ denotes complex conjugate, find H for this vortex pair and hence deduce th path of a single vortex in the half-plane $\operatorname{Im}\zeta > 0$.

(c) The quantity H remains a conserved quantity under conformal mappings but transforms through

$$H_z(z_1, z_2, \ldots, z_n) = H_\zeta(\zeta_1, \zeta_2, \ldots, \zeta_n) - (1/4\pi)\sum_{i=1}^{n}\kappa_i^2 \log|f'(z_i)|,$$

for $\zeta = f(z)$.

Use this result with that of (b), and the transformation $\zeta = z^2$ to obtain the path of a single vortex in the quarter-plane $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$ and hence sketch it.

3. An equation for the delayed oscillator model of the ENSO cycle is

$$T_t \;=\; -\alpha_1 T \;-\; (\alpha_2/\lambda)\tanh\left[\lambda(T - rT(t-\Delta)\right] \;, \qquad (1)$$

where α_1, α_2, λ, r and Δ are positive constants, with $r < 1$ and $\alpha_2 > \alpha_1$, and $T(t)$ represents the sea surface temperature anomaly in an equatorial region of a bounded ocean.

Describe briefly the physical process that is responsible for the delay time Δ.

Show that when $\alpha_2 - \alpha_1 > \alpha_2 r$ there are at least three different equilibrium values for T.

With suitable scaling, an approximate non-dimensional form for (1) is

$$T_t \;=\; T \;-\; T^3 \;-\; \gamma T(t-\delta) \quad , \qquad (2)$$

where $\gamma = r\alpha_2/(\alpha_2 - \alpha_1)$ and $\delta = (\alpha_2 - \alpha_1)\Delta$.

Consider the regime with $1/2 < \gamma < 1$, in which an equilibrium value of (2) is $T_0 = (1-\gamma)^{1/2}$.

(a) If $T = T_0 + T'$, where T' is a small perturbation, derive the linearised equation

$$T'_t \;=\; (3\gamma - 2)T' \;-\; \gamma T'(t-\delta) \quad . \qquad (3)$$

(b) Show that neutral solutions of (3) oscillate with frequency σ_I given by

$$\sigma_I \;=\; \gamma[1 - (3 - 2/\gamma)^2]^{1/2} \quad , \qquad (4)$$

and that a neutral curve (which divides regions of growing and decaying solutions) in the γ-δ plane has

$$\delta = \arccos(3 - 2/\gamma)/[\gamma^2 - (3\gamma - 2)^2]^{1/2} \;. \qquad (5)$$

Deduce that on this curve $\delta \to \infty$ as $\gamma \to 1/2$ from above.

(c) By considering $\gamma = 1 - \epsilon$ where $0 < \epsilon \ll 1$, or otherwise, show that one possible solution of (5) has $\gamma\delta \to 1$ as $\gamma \to 1$. Given that $d\gamma/d\delta$ is negative on this curve, sketch the shape of the curve.

(d) By considering the behaviour of (3) for δ=0, or otherwise, discuss qualitatively the behaviour of solutions near the neutral curve.

Discuss qualitatively the expected behaviour of solutions of (2).

What is the relevance of this behaviour to the ENSO cycle?

4. Two immiscible viscous fluids (say fluid 1 and fluid 2) are in planar motion, touching each other at their common interface, the fluid densities being ρ_1, ρ_2 and viscosities μ_1, μ_2 respectively.

   (a) Show that the governing equations in fluid 1 may be written in non-dimensional form as

   $$\text{div}\ \mathbf{u} = 0, \quad \partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\mathbf{grad})\mathbf{u} = -\mathbf{grad}\ p + \text{Re}^{-1}\boldsymbol{\Delta}\mathbf{u}$$

   where $\mathbf{u} = (u, v)$ is the velocity vector and p is the pressure. Define the Reynolds number Re.

   (b) Give the corresponding equations in fluid 2 subject to the *same* non-dimensionalisation as in (a). Also give the conditions at the interface.

   (c) If only a small thin layer of fluid 2 is present, with aspect ratio $h \ll 1$, provide a justification for the expansions

   $$(u, v, p) = (u_1, v_1, h^{-1}p_1) + \ldots \quad \text{and} \quad (u, v, p) = (h^{-1}u_2, v_2, h^{-1}p_2) + \ldots$$

   in fluids 1, 2 respectively. Provide also expressions for the length and time scales.

   (d) If the ratios ρ_2/ρ_1 and μ_2/μ_1 are small and comparable, derive from (c) the different systems of equations that can arise within fluid 1 and within the thin layer of fluid 2 over different Re ranges. Identify a critical Re range.

5. A one-dimensional model of viscous blood flow in an artery is governed by the following system of equations

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} &= 0, \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z}\left(\frac{Q^2}{A}\right) &= -\frac{A}{\rho}\frac{\partial P}{\partial z} - K\frac{Q}{A}, \\ \text{and} \quad P - P_{\text{ext}} &= \beta\left(\frac{\sqrt{A}-\sqrt{A_0}}{A_0}\right). \end{aligned}$$

Here, $Q(z,t)$, $A(z,t)$ and $P(z,t)$ are the volume flux, the cross sectional area of the artery, and the fluid pressure respectively, at a distance z along the artery and at time t. All other physical parameters present in the model remain constant.

(a) By eliminating the pressure $P(z,t)$, convert the system into the following form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{H}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial z} + \mathbf{B}(\mathbf{u}) = 0, \qquad (1)$$

where

$$\mathbf{u} = \begin{bmatrix} A \\ Q \end{bmatrix}, \quad \mathbf{H}(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ \left(\frac{\beta A^{1/2}}{2\rho A_0} - \frac{Q^2}{A^2}\right) & \frac{2Q}{A} \end{bmatrix}, \quad \text{and} \quad \mathbf{B}(\mathbf{u}) = \begin{bmatrix} 0 \\ K\frac{Q}{A} \end{bmatrix}.$$

(b) Obtain the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{H}(\mathbf{u})$ and, subsequently, determine a matrix of left eigenvectors $\mathbf{L}$ that satisfies

$$\mathbf{L}\,\mathbf{H}(\mathbf{u}) = \Lambda\,\mathbf{L} \quad \text{for } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

(c) By multiplying equation (1) by $\xi(Q,A)\,\mathbf{L}$, for some smooth scalar function ξ, derive the Riemann invariants

$$\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \frac{Q}{A} + 4\sqrt{\frac{\beta}{2\rho A_0}}A^{1/4} \\ \frac{Q}{A} - 4\sqrt{\frac{\beta}{2\rho A_0}}A^{1/4} \end{bmatrix}.$$

(d) Hence or otherwise, show that these Riemann invariants satisfy the equations

$$\left(\frac{\partial}{\partial t} + \left(\frac{Q}{A} \pm \sqrt{\frac{\beta}{2\rho A_0}}A^{1/4}\right)\frac{\partial}{\partial z}\right)\left(\frac{Q}{A} \pm 4\sqrt{\frac{\beta}{2\rho A_0}}A^{1/4}\right) + K\frac{Q}{A^2} = 0.$$