UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sc.

Financial Mathematics

COURSE CODE : **MATHG508**

DATE : **11-MAY-06**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

06-N0130-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

NOTE: In the questions which follow the current price of an asset (or similar instrument) will often be denoted either by S_t or simply by S with the time subscript suppressed. Reference is made to the following definitions:

$$
\begin{aligned}
(x)^+ &= \max\{x, 0\}, \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{t^2}{2}\right\} dt, \\
n(x) &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}, \\
d_1 &= \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \\
d_2 &= \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},
\end{aligned}
$$

where K denotes the exercise price, r the riskless rate, σ the volatility and t is the time to expiry.
The Black-Scholes formula for pricing a European call is:

$$C = SN(d_1) - Ke^{-rt}N(d_2).$$

1. (a) Explain how covered interest-rate arbitrage can be used to value a forward $F(T)$ to time T on a foreign exchange rate S. Write a formula for the value of this forward using continuously-compounded interest rates.

   (b) In the context of a one-period multi-state model of asset prices define what is meant by *arbitrage opportunity* and *risk-neutral measure.* State and prove the No-Arbitrage Theorem. You may assume the Separating Hyperplane Theorem but this must be stated carefully.

   (c) Explain the benefits to a trader of using risk-neutrality over expectation-based pricing.

2. Consider the following model, with $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	10	14	18
ω_2	10	14	12
ω_3	10	8	12
ω_4	10	8	4

(a) Replicate the call option $X = (S(2) - 7)^+$ over the two periods and so find the fair price of the claim.

(b) Find all the one period risk-neutral probabilities and the corresponding probability on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $E_{\mathbf{Q}}[X]$ is the fair price.

(c) For the same model, find the value of the following so-called *Asian option*

$$A = [\frac{1}{3}\{S(0) + S(1) + S(2)\} - 7]^+.$$

(d) In the T-period binomial model, if the asset price is S at any time, the next period's price will be either SU or SD. The interest rate per period r is positive and $D^* < 1 < U^*$, where the star denotes discounting.

(i) Describe the risk-neutral measure $\mathbf{Q}$.

(ii) A *digital option* pays one dollar at time $t = T$ if the asset price is above a fixed level K and is worthless otherwise. Using $\mathbf{Q}$ show that the option value at time $t = 0$ is equal to

$$\frac{1}{(1+r)^T} \sum_{n \geqslant \hat{n}} \binom{T}{n} q_U^n q_D^{T-n}$$

for some $\hat{n}$ which you must find.

3. (a) Let Ω be a finite set, and let $\mathbf{P}$ be a probability measure on Ω. Define what is meant by a *filtration* $P_t : t = 0, \ldots, T$ on Ω. When is a process $S(t)$ said to be *adapted* to the filtration, and when is it a *martingale*? When is a process $H(t)$ *previsible* with respect to a filtration?

(b) Give a brief explanation of the idea behind dynamic programming as applied to the valuation of an American option. Use the method to value an American call option with exercise price $K = 7$ dollars written on an asset where the asset prices in dollars are given below, the interest rate per period is zero, and a dividend of two dollars is paid between time 1 and expiry.

	$S(0,\omega)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	10	14	16
ω_2	10	14	10
ω_3	10	8	10
ω_4	10	8	2

(c) Construct a hedging strategy for the American option.

4. (a) Let $f(S,t)$ be a function of two variables (continuously twice differentiable in S and once in t). State Itô's Formula for $df(S(t),t)$, where $S(t)$ is an asset price obeying the stochastic equation

$$dS = \mu dt + \sigma dW$$

in which $W = W(t)$ is standard Brownian motion and μ, σ are continuous functions of S and t. Give a plausability argument in support of the formula.

(b) What form does Itô's Formula take when the function f is independent of time? Using this formula, explain how we can obtain a relationship between the stochastic integral and a standard integral.

(c) Find an expression for

$$\int_0^T W(t)dW(t)$$

(d) Now assume that S is a model for stock prices obeying the stochastic equation

$$dS = \mu S dt + \sigma S dW$$

What are the mean and variance of the risk-neutral probability of S given its value $S(t)$ at time t?

5. (a) Let $V(S,t)$ denote the value at time $t \leq T$ of a European option when the price of the underlying asset is S. Assume that the asset price process $S(t)$ follows the stochastic equation

$$dS = \mu S dt + \sigma S dW$$

where $W = W(t)$ is a standard Brownian motion, μ, σ are constants and r is a constant riskless interest rate applicable throughout the life of the option.

Use Itô's Formula to derive the Black-Scholes equation satisfied by the function $V(S,t)$, namely

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

(b) [Refer to the formulae at the start of the exam paper]

Show that $d_2^2 = d_1^2 - 2log(Se^{rt}/K)$. Hence, or otherwise, show that the *delta* of a European call option is

$$\frac{\partial C}{\partial S} = N(d_1).$$

What does the buyer of a European call option need to do today to hedge the exposure to the underlying stock?