UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

M.Sc. *PG Dip*

Computational and Simulation Methods

COURSE CODE : **MATHGM04**

DATE : **10-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-N0134-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the following initial value problem for a first-order ordinary differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t), \quad y(0) = y_0.$$

   (a) State the key advantages of explicit Runge-Kutta methods over

      (i) Taylor-series methods;

      (ii) multi-step methods.

   (b) Show, by considering a Taylor Series expansion of $y(t_i + h)$, that an explicit Runge-Kutta algorithm of the form

$$y_{i+1} = y_i + h\omega_1 f(y_i, t_i) + h\omega_2 f\left[y_i + \alpha h f(y_i, t_i), t_i + \beta h\right]$$

   is second-order accurate so long as certain constraints on the parameters ω_1, ω_2, α and β are satisfied. State clearly what these constraints are.

   (c) Define the growth factor g for a general numerical scheme $y_{i+1} = \tau(y_i, t_i)$. Explain why g is related to the stability of the numerical scheme and show that

$$g \approx \frac{\partial \tau}{\partial y_i}.$$

   (d) Examine the stability of the above Runge-Kutta scheme for the case $f(y) = \lambda y$. Show that the scheme is stable only for the region of the complex plane where

$$\left|1 + \lambda h + \frac{(\lambda h)^2}{2}\right| < 1.$$

   Hence, given a positive real stepsize h, for what real values of λ is the numerical scheme stable?

2. Consider the Black-Scholes pricing problem for the value of a European Call Option $V = V(S,t)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\, S \frac{\partial V}{\partial S} - rV = 0,$$

$$\begin{aligned} V(0,t) &= 0, \text{ and } \lim_{S\to\infty} V(S,t) \to S, \\ V(S,T) &= \max(S - E, 0), \end{aligned}$$

where $S \geqslant 0$ is the spot price of the underlying financial asset, $0 < t \leqslant T$ is the time, $E > 0$ is the strike price, $T > 0$ is the expiry date, $r \geqslant 0$ is the interest rate, D is the dividend yield, and σ is the volatility of S.

(a) By writing $S = n\delta S$ for $0 \leqslant n \leqslant N$, $t = m\delta t$ for $0 \leqslant m \leqslant M$, and expressing the derivative terms as

$$\frac{\partial V}{\partial t} \sim \frac{V_n^m - V_n^{m-1}}{\delta t}, \quad \frac{\partial V}{\partial S} \sim \frac{V_{n+1}^m - V_{n-1}^m}{2\delta S}, \quad \frac{\partial^2 V}{\partial S^2} \sim \frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\delta S^2},$$

(you are not required to derive the Taylor series expansions) obtain the *backward marching* scheme in time,

$$V_n^{m-1} = \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m, \qquad \text{(BMS)}$$

where

$$\alpha_n = \frac{1}{2}\left(n^2\sigma^2 - n(r - D)\right)\delta t, \ \beta_n = 1 - \left(r + n^2\sigma^2\right)\delta t, \ \gamma_n = \frac{1}{2}\left(n^2\sigma^2 + n(r - D)\right)\delta t.$$

(b) Show that the payoff and boundary conditions, in turn, can be expressed in finite difference form as

Final Payoff:	$V_n^M = \max(n\delta S - E, 0) \quad 0 \leqslant n \leqslant N;$
At $S = 0$:	$V_0^{m-1} = \beta_0 V_0^m \quad M \geqslant m \geqslant 1;$
As $S \to \infty$:	$V_N^{m-1} = (\alpha_N - \gamma_N)\, V_{N-1}^m + (\beta_N + 2\gamma_N)\, V_N^m \quad M \geqslant m \geqslant 1.$

(c) Consider an initial disturbance that is proportional to $\exp(in\omega)$. If $\widehat{V}_n^m$ is an approximation to the exact solution ${V_n}^m$, then

$$\widehat{V}_n^m = {V_n}^m + {E_n}^m,$$

where ${E_n}^m$ is the associated error, and $\widehat{V}_n^m$ also satisfies (BMS) to give

$$E_n^{m-1} = \alpha_n E_{n-1}^m + \beta_n E_n^m + \gamma_n E_{n+1}^m.$$

By putting

$${E_n}^m = \lambda^m \exp(in\omega),$$

which is an oscillatory expression of amplitude λ^m and frequency ω, use a Fourier stability analysis to show that for this scheme to remain stable requires the strict condition: $\delta t \sim O(N^{-2})$.

3. (a) The Excel function RAND() produces a random variable which is uniformly distributed over 0 and 1. Show that this has a mean $\mu = \frac{1}{2}$ and variance $\sigma^2 = \frac{1}{12}$.

(b) If we generate any number N of these random variables then, by the Central Limit Theorem, the algorithm,

$$\sqrt{\frac{12}{N}}\left(\sum_{1}^{N} RAND() - \frac{N}{2}\right), \qquad \text{(RV)}$$

produces a single standardised Normal $\phi \sim N(0,1)$, i.e. each ϕ is normally distributed with mean zero and variance of one. Show that the expression given by (RV) does produce a Normally-distributed random variable with zero mean and unit variance.

(c) The fair price of an option is defined as the "expected value of the discounted payoffs under the risk-neutral measure", i.e.

$$\mathbb{E}_{\mathbb{Q}}\left(\left[\exp - \int_t^T r(\tau)\, d\tau\right] \text{Payoff}(S)\right)$$

where $S(t)$ is an asset price, $r(t)$ is the interest rate and $\mathbb{Q}$ is the risk-neutral density. A European call option is to be priced written on an equity using stochastic interest rates. Suppose this stock price S evolves according to the lognormal random walk and the interest rate follows the Cox-Ingersoll-Ross model. The increments in Brownian Motion dX_i of the two processes are correlated such that $\mathbb{E}[dX_1 dX_2] = \rho dt$, where ρ is the correlation coefficient and dt the time step.

Describe in detail the Monte Carlo scheme you would use to price such a contract, which should include details of how to:

(i) discretise the relevant stochastic differential equations;

(ii) produce correlated random variables;

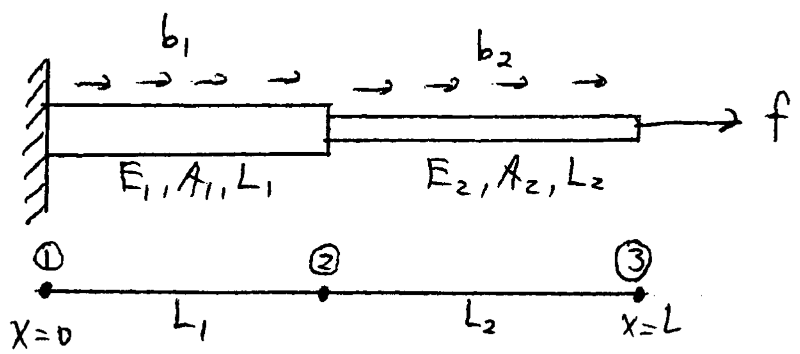
(iii) calculate the discount factor.

4. An axially loaded elastic bar consists of two parts, 1 and 2, of length L_1 and L_2, each having a constant Young modulus E, constant cross-sectional area A and constant external body force per unit axial length b (see the figure below). The equation for the axial displacement y of the bar is given by

$$\frac{d}{dx}\left(EA\frac{dy}{dx}\right) + b = 0, \qquad 0 \leqslant x \leqslant L,$$

where L is the total length of the bar. The bar is fixed at its left end ($x = 0$), while the right end is subjected to a tensile force f; *i.e.*, we have the boundary conditions

$$y(0) = 0, \qquad \left(EA\frac{dy}{dx}\right)_{x=L} = f.$$



(a) Derive the weak formulation of the boundary-value problem.

(b) Using the Galerkin approach, deduce that the single-element stiffness matrix, K^e, and load vector, f_l^e, are given by

$$K_{ij}^e = \int_{x_k}^{x_l} \left(EA\frac{dN_i^e}{dx}\frac{dN_j^e}{dx}\right) dx, \qquad f_{l\,j}^e = \int_{x_k}^{x_l} bN_j^e \, dx,$$

with N_i^e ($i = 1, 2$) the shape functions of a 2-node linear element, and x_k and x_l the element boundaries.

(c) Solve the FE equation for a mesh of two simple linear elements of length L_1 and L_2.

(d) Obtain the tensile force at $x = 0$ required to maintain equilibrium of the bar.

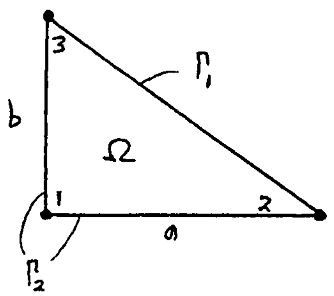
5. Consider Poisson's equation,

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f,$$

on the triangular domain Ω shown, subject to the boundary conditions

$$\begin{cases} u = g & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2 \end{cases} \qquad \Gamma_1 \cup \Gamma_2 = \Gamma(\Omega).$$

Here, $\frac{\partial u}{\partial n}$ is the normal derivative, and f and g are given functions.



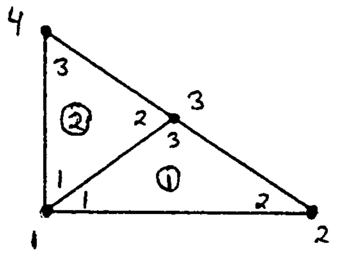
(a) Derive the weak formulation of this boundary-value problem.

(b) Compute the finite-element stiffness matrix for this problem using a single element. The shape functions for a simple 3-node triangular element are given by

$$N_i^e(x, y) = \frac{1}{2A}\left[x_j y_k - x_k y_j + (y_j - y_k)x + (x_k - x_j)y\right] \qquad (i, j, k \text{ cyclic}),$$

where (x_i, y_i) are the coordinates of the ith node and A is the area of Ω.

(c) The triangular domain is now subdivided into two 3-node triangular finite elements (see below). The local and global node numbering is as indicated. Find the global stiffness matrix in terms of the entries K_{ij}^1 and K_{ij}^2 of the element matrices K^1 and K^2 (do not actually compute these element matrices).



6. The concentration of a tracer chemical, $q(x,t)$, in a thin tube satisfies the one-dimensional conservation law,

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) = 0, \qquad \text{(CL)}$$

for some given flux function $f(q)$.

(a) By integrating (CL) over a finite-volume cell $C_i = (x_{i-1/2}, x_{i+1/2})$, and then integrating from time $t = t_n$ to $t = t_{n+1}$, obtain the finite-volume form,

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right),$$

where Q_i^n is the average value of $q(x,t)$ over the cell C_i at time t_n, $\Delta x = (x_{i+1/2} - x_{i-1/2})$ and $\Delta t = (t_{n+1} - t_n)$. Define the integral forms of Q_i^n and $F_{i\pm 1/2}$ in terms of q and $f(q)$.

(b) Suppose the following explicit finite-volume method is used to solve (CL):

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}\left(Q_i^n, Q_{i+1}^n\right) - \mathcal{F}\left(Q_{i-1}^n, Q_i^n\right)\right].$$

Here, $\mathcal{F}$ is a numerical approximation to the flux function that depends **only** on the neighbouring cell averages at the previous time step, which are Q_{i-1}^n, Q_i^n and Q_{i+1}^n. By taking, as a specific example, the advection equation

$$\frac{\partial q}{\partial t} + \bar{u}\frac{\partial q}{\partial x} = 0,$$

for a given positive constant $\bar{u}$, explain graphically why the Courant, Friedrichs and Lewy (CFL) Condition that

$$\left|\frac{\bar{u}\Delta t}{\Delta x}\right| \leqslant 1$$

is a necessary condition for the stability of such a method.

(c) The Lax-Friedrichs finite-volume method is given by

$$Q_i^{n+1} = \frac{1}{2}\left(Q_{i-1}^n + Q_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left[f\left(Q_{i+1}^n\right) - f\left(Q_{i-1}^n\right)\right]. \qquad \text{(LF)}$$

Show that this method is equivalent to a finite-difference approximation to the advection-diffusion equation,

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(f(Q)\right) = \beta\frac{\partial^2 Q}{\partial x^2},$$

where $\beta = \frac{\Delta x^2}{2\Delta t}$. *Hint: use forward differencing in time and central differencing in space.* Hence, explain what problems may arise on using the Lax-Friedrichs method (LF) to solve the conservation law (CL)?

 END OF PAPER