UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

M.Sc. *PG Dip*

Advanced Modelling Mathematical Techniques

COURSE CODE : **MATHGM01**

DATE : **02-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-N0131-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. In all of the following B is as standard Brownian motion, and all stochastic differential equations are assumed to be of Itô type.

   (a) Find the stochastic differential equation for $X_t = \exp(\alpha B_t - \beta t)$, where α and β are constants, and find the relationship between α and β that makes the drift term exactly zero.

   (b) Suppose that $X_t = \sigma B_t + ut$, where σ and u are constants. Suppose that X starts at $x \in [0, 1]$ at time $t = 0$, and is stopped at the first time T where either $X_T = 0$ or $X_T = 1$. Find $\phi(x) = \mathrm{P}^x[X_T = 1]$ by first finding an ordinary differential equation (and boundary conditions) for ϕ, and then solving this equation.

   (c) Consider ϕ of part 1b, and find

$$\lim_{u \to 0} \phi(x), \qquad 0 \leqslant x \leqslant 1.$$

2. Consider the motion of a diffusing particle, starting at the origin at time $t = 0$, in a two-dimensional shear flow governed by the following coupled Itô stochastic differential equations:

$$dX_t = \gamma Y_t dt + \sigma dB_t, \qquad dY_t = \sigma dW_t,$$

   where γ and σ are constants, B and W are independent standard (one-dimensional) Brownian motions, and (X_t, Y_t) is the particle position. Find the following expectation:

$$\mathrm{E}[Y_t], \qquad \mathrm{E}[X_t], \qquad \mathrm{E}[Y_t^2], \qquad \mathrm{E}[X_t^2], \qquad \mathrm{E}[X_t Y_t].$$

3. Consider the following differential equation for $f(x)$:

$$\varepsilon\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)^3+\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2+2f=0,$$

in which ε is a small positive parameter, $\varepsilon \ll 1$.

Find the scalings $f=\varepsilon^\alpha F$ and stretches $x=a+\varepsilon^\beta z$ at which two dominant terms balance, and sketch these balance scalings in the α–β plane. Hence determine the critical scaling and stretching at which all three terms balance.

Show that the function

$$g(y)=\cos y+\sin y$$

satisfies the ordinary differential equation

$$\left(\frac{\mathrm{d}^2 g}{\mathrm{d}y^2}\right)^3+\frac{\mathrm{d}^2 g}{\mathrm{d}y^2}\left(\frac{\mathrm{d}g}{\mathrm{d}y}\right)^2+2g=0,$$

and deduce an exact solution to the original equation for $f(x)$.

If we are constrained by the boundary conditions to have $\alpha=-2$, what two values are possible for β?

4. A function $f(x)$ satisfies the following differential equation:

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}+\frac{\mathrm{d}f}{\mathrm{d}x}+f^2=1,$$

with boundary conditions $f(0)=0$, $f(1)=1$. The parameter ε is small and positive, $\varepsilon \ll 1$.

Find an exact solution to the governing equation which satisfies the boundary condition at $x=1$.

Assume that there is a boundary layer near $x=0$. How does the size of this layer scale with ε? The solution in this "inner" region, satisfying the boundary condition at $x=0$, can be expressed as an expansion in ε. Calculate the first two terms of this expansion.

Match your two expressions to determine any unknown constants.

5. (a) Let $z = x + iy$ and $\bar{z} = x - iy$. Show that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}.$$

Find also a similar expression for $\frac{\partial}{\partial y}$.

Let $f(x, y) = u(x, y) + iv(x, y)$, where u and v are real functions. State the Cauchy-Riemann equations in terms of partial derivatives of u and v, with respect to x and y. Show that they are equivalent to $\partial f / \partial \bar{z} = 0$.

(b) Show that

$$\nabla^2 = 4 \frac{\partial^2}{\partial z \partial \bar{z}},$$

where ∇^2 is the Laplacian operator. Let $w = f(z)$ be an analytic function of z which conformally maps domain D in the z-plane to domain Δ in the w-plane. Show that ϕ satisfies Laplace's equation in D if, and only if, it satisfies Laplace's equation in Δ.

(c) Steady temperature $T(x, y)$ satisfies Laplace's equation in a two-dimensional region consisting of two semi-infinite plates $|x| \geq 1$, $y = 0$. The right-hand plate has temperature $T = 0$ and the left-plate has temperature $T = 1$ (see figure). Letting $z = x + iy$, find the images of the points A,B,C,D,E,F under the map $w = z + \sqrt{z^2 - 1}$, where the sign of the square root is chosen so that it has positive imaginary part. *Note that points B and D correspond to $z = -1$ and $z = +1$ respectively, and the other points are at $\Re z = \pm\infty$ either just above or below the plates.*

Deduce that the region in the z-plane maps to the upper half of the w-plane. Solve Laplace's equation in the w-plane and show that between along the straight line segment BD

$$T(x) = \frac{1}{\pi} \tan^{-1} \left(\frac{\sqrt{1 - x^2}}{x} \right).$$

6. (a) Define what is meant by the Schwarz function $S(\zeta)$ for a curve ∂D in the complex ζ-plane. Find the Schwarz functions for (i) unit circle centred at the origin and (ii) the imaginary axis.

(b) The map $z = \alpha\zeta + \beta/\zeta$, $\alpha, \beta \in \Re$, $0 < \beta < \alpha$, maps the exterior of the unit circle centered at the origin in the ζ-plane to the exterior of an ellipse in the z-plane. The ellipse has major axis of length $2a$ aligned with the real z axis and minor axis of length $2b$ aligned with the imaginary z axis, where $a = \alpha + \beta$ and $b = \alpha - \beta$.

Show that the Schwarz function for the ellipse in the z-plane is given by

$$S(z) = \frac{ab}{\alpha\zeta} + \frac{\beta}{\alpha}z.$$

The ellipse represents the boundary of a patch D of uniform vorticity with unit magnitude. The ellipse is observed to be steady when placed in a uniform strain field having velocity components $u_E = -\epsilon y$ and $v_E = -\epsilon x$. Consider the velocity field given by

$$u - iv = \begin{cases} -\frac{i}{2}(\bar{z} - F(z)) + i\epsilon z, & z \in D \\ -\frac{i}{2}G(z) + i\epsilon z, & z \notin D \end{cases} \tag{1}$$

where $G(z) = ab/(\alpha\zeta)$ is an analytic function outside D and $F(z) = S(z) - G(z)$ is analytic inside D.

Given that the vorticity is $\omega = v_x - u_y$, show that if $u - iv = A(z)$, where $A(z)$ is any analytic function, then the vorticity is zero (i.e. irrotational flow). Hence show that the velocity field (1) is irrotational outside D and has unit magnitude inside D. Further, show that $u - iv$ is continuous on ∂D and tends to the uniform strain field at large distance from the vorticity patch.

By demanding that on ∂D the velocity $u + iv$ be parallel to the boundary of the patch show that

$$\epsilon = \frac{ab(a-b)}{(a+b)(a^2+b^2)}.$$

[*Hint: the boundary ∂D can be parameterised as $z(\theta)$ where $\zeta = e^{i\theta}$, $0 \le \theta < 2\pi$.*]