# UNIVERSITY COLLEGE LONDON

University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

*B.Sc.* *M.Sci.*

**Mathematics C353: Theory Of Traffic Flow I**

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| COURSE CODE | : | **MATHC353** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **28-APR-06** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

06-C1016-3-60


**TURN OVER**

*All questions may be answered, but only marks obtained on the best **four** questions will count.*
*The use of an electronic calculator is **not** permitted in this examination.*

1 You may assume that for any strictly positive $a_i$ $(i = 1, 2, \dots, I)$ and $b_j$ $(j = 1, 2, \dots, J)$ such that $\Sigma_i a_i = \Sigma_j b_j$, any real $I \times J$ matrix $(c_{ij})$ and any real $\alpha$ there is an $I \times J$ matrix $\mathbf{t}^*(\alpha)$ with row and column sums $a_i$ and $b_j$ and such that for all $i$ and $j$, $t^*_{ij} = r_i s_j \exp(-\alpha c_{ij})$ for some $r_i$ and $s_j$.

Let $D$ be the set of all $I \times J$ matrices $\mathbf{t} = (t_{ij})$ having row sums $a_i$ $(i = 1, 2, \dots, I)$ and column sums $b_j$ $(j = 1, 2, \dots, J)$ and such that $t_{ij} > 0$ for all $i$ and $j$.

For $\mathbf{t}$ in $D$, let $F(\mathbf{t}) = \Sigma_{ij}\, t_{ij} \ln t_{ij} + \alpha\, \Sigma_{ij}\, c_{ij}\, t_{ij}$.

(a) Show that for $\mathbf{t}$ in $D$ the matrix of second derivatives of $F(\mathbf{t})$ with respect to the $IJ$ variables $t_{ij}$ is positive definite.

(b) State without proof two consequences of (a) concerning stationary points of $F$ in $D$.

(c) By constructing and differentiating a suitable Lagrangian function, show that $\mathbf{t}^*(\alpha)$ is a stationary point of $F$ in $D$.

Use the Furness procedure to calculate $\mathbf{t}^*(0)$ when $I = 2$, $J = 3$, $a_1 = a_2 = 600$, $b_1 = 300$, $b_2 = 400$ and $b_3 = 500$.

The $c_{ij}$ are costs of travel and the $t_{ij}$ are numbers of journeys between origins $i$ and destinations $j$ in a city. State briefly the interpretation of the two terms in the function $F(\mathbf{t})$. How would you expect the relevant value of $\alpha$ to change over, say, 20 years during which the average income of the inhabitants of the city increased substantially?

2 Wardrop's first principle of route choice is that "the journey times on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route."

In a network in which traffic respects the first-in first-out (FIFO) discipline, the demand for travel varies over time. Show that, in order for traffic flows in this network to satisfy Wardrop's first principle, the proportion $\mu_p(s)$ of demand that is assigned to route $p$ at time $s$ satisfies

$$\mu_p(s) = \frac{g_p[\tau(s)]}{\sum_{q \in P} g_q[\tau(s)]} \quad \forall p \in P,$$

where $\tau(s)$ is the time of arrival of traffic that departs at time $s$,

$g_p(t)$ is the outflow from route $p$ at time $t$,

$P$ is the set of routes available for that journey.

Discuss the use in the right-hand side of this expression of route outflows at time $\tau(s)$ to calculate assignment proportions at time $s < \tau(s)$.

3 Customers arrive at a queue for a certain facility according to a Poisson process with mean rate $q$ and have mutually independent service times exponentially distributed with mean $s^{-1}$. If the facility is occupied when a customer arrives, then the customer goes elsewhere; otherwise the customer occupies the service facility immediately. Show that the probability of the facility being occupied at time $t \geq 0$ is given by

$$P_B(t) = [1 + \exp\{-(q+s)t\}]\left(\frac{q}{s+q}\right) + \exp\{-(q+s)t\} P_B(0) .$$

Show that this probability changes less rapidly as time increases. Hence or otherwise show that the rate of change of mean occupancy of the service facility lies in the range $[-s, q]$, and identify a case in which each of the extreme values $-s$ and $q$ is attained.

4 (a) Explain what is meant by each of a *shock wave* and a *wave* in traffic, and establish an expression for the speed at which each of these travels.

The flow of a stream of traffic is interrupted between times $t=0$ and $t=r$ by the effective red period of a traffic signal. At all times, the traffic approaches the signal freely at rate $q$ and speed $v$, and after time $t=r$ the signal remains green indefinitely.

Show that the trajectory of $x_b$, the back of the queue of stationary traffic, initially satisfies

$$x_b = \frac{-qlvt}{(ql-v)},$$

where $l$ is the effective length of a queued vehicle.

Show that the flow $q_r(k)$ past a wave of density $k$ satisfies

$$q_r(k) = -k^2 \frac{dv}{dk}.$$

Using the variable $s$, the saturation flow, derive an expression for the time at which traffic conditions at the stop-line return to normal.

4 (b) At a signal-controlled road junction there are two streams of traffic, each having green in one of the two stages of the signal cycle The cycle time must not exceed $c_0$ and proportions $\lambda_1$ and $\lambda_2$ of the cycle are effectively green for Stages 1 and 2 respectively. The lost time per cycle is $L$, the flow ratios in Streams 1 and 2 are $y_1$ and $y_2$ respectively and their maximum acceptable degrees of saturation are $p_1$ and $p_2$ respectively. No minimum green constraints are imposed.

The arrival rates in the two streams are multiplied by a common factor $\mu$. Derive the equations for the three planes in $(\lambda_1, \lambda_2, \mu)$ space that form (together with the plane $\mu = 0$) the boundaries of the region containing acceptable values of $(\lambda_1, \lambda_2, \mu)$.

Hence find the coordinates of the vertex of this region at which $\mu$ is largest.

Corresponding to this vertex, what names are given to the signal timings and the conditions under which the junction is operating?

5 At a signal-controlled road junction there are $m$ stages in the signal cycle and $n$ streams of traffic. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the effective green times for Stage $i$ and Stream $j$ form proportions $\lambda_i$ and $\Lambda_j$ of the cycle respectively. The lost time forms a proportion $\lambda_0$ of the cycle, and $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2 \ldots, \lambda_m)$.

The $\Lambda_j$ are known linear combinations of the components of $\boldsymbol{\lambda}$, and the flow ratio of each stream $j$ has a known value $y_j$.

What is the value of the sum of the components of $\boldsymbol{\lambda}$ and why is this so?

Express as linear inequalities in the components of $\boldsymbol{\lambda}$ the usual practical constraints on the cycle time and the durations of the stages.

Provided that $\Lambda_j > y_j$ for all $j$, the average delay per unit time to vehicles at the junction is approximately proportional to

$$D(\boldsymbol{\lambda}) = \Sigma_j \{f_j(\Lambda_j)/\lambda_0 + g_j(\Lambda_j)\},$$

where $f_j(\Lambda) = Lq_j(1-\Lambda)^2/2(1-y_j)$ and $g_j(\Lambda) = y_j^2/2\Lambda(\Lambda - y_j)$.

$S$ is the set of values of $\boldsymbol{\lambda}$ such that the above constraints on the components of $\boldsymbol{\lambda}$ and on the $\Lambda_j$ are satisfied. One member $\boldsymbol{\lambda}^*$ of $S$ is known.

Show that there is a member $\hat{\boldsymbol{\lambda}}$ of $S$ such that $D(\boldsymbol{\lambda}) \geq D(\hat{\boldsymbol{\lambda}})$ for all $\boldsymbol{\lambda}$ in $S$.

For a junction at which $m = 2$, use the equation for the sum of the components of $\boldsymbol{\lambda}$ to express the cycle time and stage duration constraints in terms of $\lambda_1$ and $\lambda_2$ only. Hence sketch the boundaries that $S$ would have in the $(\lambda_1, \lambda_2)$ plane if traffic was very light in every stream.

Stream 1 has green only in Stage 1 and none of the lost time is effectively green for this stream. The traffic in Stream 1 is heavy enough to add another boundary to $S$. Add to your sketch a line representing the constraint $\Lambda_1 > y_1$.