UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

B.Sc.

Mathematics C353: Theory Of Traffic Flow I

COURSE CODE	:	**MATHC353**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0932-3-30

TURN OVER

All questions may be answered, but only marks obtained on the best ***four*** *questions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1 Customers arrive at a queue according to a Poisson process with mean rate q and have exponentially distributed service times with mean Q^{-1}. Show that if $x = q/Q < 1$ then the equilibrium probability P_n that there are n customers in the queue is given by

$$P_n = (1-x)x^n \quad (n \geq 0).$$

Hence or otherwise show that $\mathrm{E}(N)$, the mean equilibrium number of customers in the queue, is given by

$$\mathrm{E}(N) = \frac{x}{(1-x)}$$

and that the probability $\mathrm{P}(N > 0)$ of the queue being non-empty is equal to x. Discuss the behaviour of this queueing system as the arrival rate q varies in $[0, Q)$.

2 For a stream of traffic controlled by fixed-time signals, the arrival rate is q, the saturation flow is s, a proportion Λ of the cycle time c is effectively green, and $q < \Lambda s$. A cycle starts at the beginning of effective red with no traffic queueing. Show that:

(a) if arrivals are uniform then the delay incurred in this cycle is $A_u = qc^2(1 - \Lambda)^2 s/2(s - q)$; and

(b) if the amount of traffic arriving in any interval of duration t is the random variable Q_t and if the effective green in this cycle were extended at least until the queue of traffic had cleared, then the expectation of the delay incurred in this cycle would exceed A_u.

This stream and a second identical one are the only traffic streams at a junction where the signal cycle has two stages, one stream has green in each stage, the lost time per cycle is L and the maximum cycle time is c_0, where $L < c_0(1 - 2q/s)$. When all the effects of randomness in arrivals are taken into account, the average delay per unit time at the junction is estimated by $9D/10$, where $D = 2A_u/c + q^2/s\Lambda(s\Lambda - q)$. The signals are timed to give each stage effective green for a separate proportion λ of the cycle, and the lost time forms the remaining proportion $1 - 2\lambda$. Express c and Λ in terms of λ and deduce the range of values λ is permitted to take.

By examining the behaviour of the two terms of D over this range, show that the value of λ that minimises D over this range can lie either inside the range or at its upper endpoint.

3 (a) A signal-controlled road junction has the same number m of traffic streams as there are stages in the signal cycle, and just one stream has green in each stage. The maximum cycle time is c_0, the lost time per cycle is L, and in each stream i the arrival rate is μq_i, the saturation flow is s_i and the maximum acceptable degree of saturation is p_i. No minimum green times are specified.

For this particular case, obtain and solve the equations for the value μ^* of μ that determines the practical capacity of the junction for arrival rates proportional to the q_i and the corresponding capacity-maximising signal timings.

It is then specified that the green time for stage 1 must be at least g_{1M}. For what values of g_{1M} will μ^* be altered?

(b) Customers arrive at a queue according to a Poisson process with mean rate q, and have independent and identical (constant) service times μ. Show that if $x = \mu q < 1$, then the mean delay d incurred by customers in this queue is given by

$$d_r = \frac{\mu}{2}\left(\frac{x}{1-x}\right).$$

Discuss the behaviour of this queue as $x \to 1$ from below and mention a use of the expression for d_r in analysing the traffic signal queue.

4 Define the two kinds of iteration in the Furness iterative procedure used to obtain from any $I \times J$ matrix $\tau = (\tau_{ij})$ whose elements are strictly positive a matrix $\mathbf{t}^* = (t_{ij}^*)$ which has the form $t_{ij}^* = r_i s_j \tau_{ij}$ for some r_i and s_j and all i and j and has given desired strictly positive row and column sums a_i $(i = 1, 2, \ldots, I)$ and b_j $(j = 1, 2, \ldots, J)$ such that $\Sigma_i a_i = \Sigma_j b_j$.

Prove that, subject to the convergence, which you may assume, of two infinite products, the procedure converges and the resulting limit matrix does indeed have the desired sums.

Suppose that, for all i and j, $\tau_{ij} = \exp(-\alpha c_{ij})$, where α is a parameter and $\mathbf{c} = (c_{ij})$ is the matrix of costs of travel from origin zone i to destination zone j in a city, and let the corresponding matrix $\mathbf{t}^*$ be denoted by $\mathbf{t}^*(\alpha, \mathbf{c})$. Use the uniqueness of matrix $\mathbf{t}^*$ for given τ to show that if $\mathbf{c}' = (c_{ij}')$ is a second matrix of costs such that $c_{ij}' = c_{ij} + u_i + v_j$ for all i and j, then $\mathbf{t}^*(\alpha, \mathbf{c}') = \mathbf{t}^*(\alpha, \mathbf{c})$.

The zones correspond to the catchment areas of local rail stations in the city, where the stations are owned by one company and the train operation by another. Comment on the relevance of the last result to a proposal by the station owners to charge passengers an extra amount, specific to each station, to enter or leave that station in addition to the train fare.

5 Discuss ways in which travellers might respond to changes in their costs of travel through a road network.

Show that an assignment with path flows **t** according to Wardrop's equilibrium principle that is consistent with trip distribution **T** according to the gravity model with exponential cost functions can be found by solving the optimisation problem:

$$\underset{\mathbf{t},\mathbf{T}}{\text{Minimise}} \sum_{a\in L} \int_{v=0}^{v_a} c_a(v)\,dv + \frac{1}{\alpha}\sum_{od} T_{od}\left(\log_e T_{od} - 1\right)$$

subject to the constraints

$$\sum_{p\in P_{od}} t_p = T_{od} \quad \forall od,$$

$$\sum_{d} T_{od} = A_o \quad \forall o,$$

$$\sum_{o} T_{od} = B_d \quad \forall d,$$

$$t_p \geq 0 \quad \forall od \ \forall p \in P_{od},$$

$$v_a = \sum_{od} \sum_{p\in P_{od}} t_p\, \delta_p^a \quad \forall a \in L,$$

where

$$\delta_p^a = \begin{cases} 1 & \text{if link } a \text{ is on path } p \\ 0 & \text{otherwise,} \end{cases}$$

$c_a(v)$ is the cost of using link a when the flow on that link is v,

L is the set of links in the network,

T_{od} is the demand for travel from o to d per unit time,

t_p is the flow per unit time on path p,

P_{od} is the set of reasonable paths from o to d, and

v_a is the flow on link a.