UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C353: Theory Of Traffic Flow I

COURSE CODE	:	**MATHC353**
UNIT VALUE	:	**0.50**
DATE	:	**10-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0927-3-40

TURN OVER

All questions may be answered, but only marks obtained on the best ***four*** *questions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1 A city is divided into I origin zones and J destination zones. The cost of travel and number of trips from origin zone i to destination zone j are c_{ij} and t_{ij} respectively ($i = 1, 2, \ldots, I; j = 1, 2, \ldots, J$). The $I \times J$ matrix (t_{ij}) is denoted by $\mathbf{t}$.

Let $D = \{\mathbf{t}: t_{ij} > 0$ and $\mathbf{t}$ has specified row sums a_i $(i = 1, 2, \ldots, I)$ and column sums b_j $(j = 1, 2, \ldots, J)\}$.

For any $\mathbf{t} \in D$, $F(\mathbf{t}) = T(\mathbf{t}) + \alpha C(\mathbf{t})$, where $C(\mathbf{t}) = \Sigma_{ij}\, c_{ij}\, t_{ij}$, $T(\mathbf{t}) = \Sigma_{ij}\, t_{ij} \ln t_{ij}$, and α is a parameter.

Show that for $\mathbf{t} \in D$, $F(\mathbf{t})$ has a matrix of second derivatives with respect to the IJ variables t_{ij} which is positive definite.

What does this imply about any stationary values of $F(\mathbf{t})$ in D?

According to a certain trip distribution model, $\mathbf{t} = \mathbf{t}^*(\alpha)$, where $t_{ij}^* = r_i s_j \exp(-\alpha\, c_{ij})$ and the r_i and s_j are chosen to make $\mathbf{t}^*(\alpha)$ a member of D.

State without proof a necessary and sufficient condition on the c_{ij} for $\alpha \neq \beta$ to imply that $\mathbf{t}^*(\alpha) \neq \mathbf{t}^*(\beta)$.

On the assumptions that this condition is satisfied and that, for any α, $\mathbf{t}^*(\alpha)$ is a stationary value of $F(\mathbf{t})$ in D, show that $C^*(\alpha) = C\{\mathbf{t}^*(\alpha)\}$ is a strictly decreasing function of α.

Calculate $C^*(0)$ when $I = 2$, $J = 3$, $a_1 = 600$, $a_2 = 400$, $b_1 = 200$, $b_2 = 300$, $b_3 = 500$ and

$$(c_{ij}) = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

2(a) Customers arrive at a queue according to a Poisson process with mean rate q, and have independent and identical (constant) service times μ. Show that if $x = \mu q < 1$, then the mean delay d incurred by customers in this queue is given by

$$d = \frac{\mu}{2}\left(\frac{x}{1-x}\right).$$

Explain why as it stands, this expression is unsuitable for use to estimate delays at a traffic signal, and describe how it can be used within one that is more suitable.

(b) Traffic in a certain stream can be regarded as a fluid arriving uniformly at rate q vehicles/second. The stream is controlled by a fixed-time traffic signal with effective red time 60 seconds at which the saturation flow is 1 vehicle/second.

Show that if the cycle time exceeds $60/(1-q)$ seconds, then the delay incurred in one cycle is $1800q/(1-q)$ vehicle-seconds.

The cycle time is 160 seconds, and owing to changing conditions at roadworks upstream of the signal, the value of q is:

0.4 in one-third of the cycles,
0.5 in another one-third of the cycles, and
0.6 in the remaining one-third of the cycles.

Show that the average delay per cycle over all cycles exceeds by 100 vehicle-seconds the delay per cycle that would occur if the same amount of traffic arrived but did so at the same rate in every cycle.

What general properties of the delay per cycle at a fixed-time signal does this result illustrate?

3 Wardrop's first principle of route choice is that "the journey times on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route".

In a network in which traffic respects first-in first-out (FIFO) discipline, the demand for travel varies over time. Show that, in order for traffic flows in this network to satisfy Wardrop's first principle, the proportion $\mu_p(s)$ of demand that is assigned to route p at time s satisfies

$$\mu_p(s) = \frac{g_p[\tau(s)]}{\sum_{q \in P} g_q[\tau(s)]} \quad \forall p \in P$$

where $\tau(s)$ is the time of arrival of traffic that departs at time s,
$g_p(t)$ is the outflow from route p at time t, and
P is the set of routes available for that journey.

Discuss the requirements on traffic models for use in this context that arise in order to achieve plausible route choice behaviour.

4 Traffic from a one-way side-road enters a two-way main road at a T-junction controlled by fixed-time traffic signals with a two-stage cycle and three streams of vehicular traffic

Define the variables λ_0, λ_1, λ_2 and μ that are used in calculating signal timings and traffic capacity for such a junction.

The total lost time per cycle, including time for pedestrians to cross both roads, is 20 seconds, none of which is effectively green for any of the vehicular traffic. The minimum green time for each stage is 10 seconds and the maximum cycle time is 100 seconds.

Traffic arrives at 600 vehicles/hour on the side road where the saturation flow is 2000 vehicles/hour, and at 1200 vehicles/hour in each direction on the main road where the saturation flows are 4000 vehicles/hour in one direction and 4200 in the other. The maximum acceptable degree of saturation in each stream is 9/10.

Leaving all ratios in rational form, write down all the usual constraints upon λ_0, λ_1, λ_2 and μ, and explain the meaning of each constraint.

(a) By assuming that capacity is maximised by choosing a cycle time of 100 seconds, calculate the practical capacity of the junction for flows proportional to those given, and the signal timings needed to achieve this capacity.

(b) Traffic on the side road is reduced to 60 vehicles/hour. Rework the calculations in (a) for this changed situation.

In which of the two traffic streams on the main road is the delay the greater, and why?

5 Explain clearly the physical interpretation and derive from first principles a formula for the speed of each of a *shock wave* and a *traffic wave*. Show that at each density, vehicles travel faster than do waves and establish an expression for the rate at which traffic passes a wave of density k.

A traffic signal is located at position $x = 0$ on a road, where x increases in the direction of travel and traffic generally moves according to the speed-density relationship

$$v = v_0\left(1 - \frac{k}{k_j}\right) \quad \left(0 \le k \le k_j\right)$$

where v_0 is the free-flow speed and k_j is the jam density. Before time $t = -r$ the signal is green and the traffic flow is uniform everywhere at rate q_a. At that time, the traffic signal changes to red so that a queue of density $k = k_j$ starts to form in the region $x < 0$. At time $t = 0$, the signal changes to green and traffic in the queue starts to move according to the speed-density relationship.

Show that at time $t = 0$ the density $k(x, t)$ of traffic at position x satisfies

$$k(x,0) = \begin{cases} k_j & \left(\dfrac{-q_a\, r}{k_j - k_a} < x < 0\right) \\ 0 & \left(0 < x < v_0\left(1 - \dfrac{k_a}{k_j}\right) r\right) \\ k_a & \text{(otherwise)} \end{cases}$$

where $k_a = \dfrac{k_j}{2}\left(1 - \sqrt{1 - \dfrac{4\,q_a}{v_0\, k_j}}\right)$.

Describe qualitatively how the traffic will develop after the signal turns green, illustrating your answer with appropriate sketch diagrams.