UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc.   *B.Sc.(Econ)M.Sci.*

Mathematics C353: Theory Of Traffic Flow I

COURSE CODE	:	**MATHC353**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0921-3-50

*All questions may be answered, but only marks obtained on the best **four** questions will count.*
*The use of an electronic calculator is **not** permitted in this examination.*

1 At a signal-controlled road junction there are m stages in the signal cycle and n streams of traffic. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the effective green times for stage i and stream j form proportions λ_i and Λ_j of the cycle respectively, and lost time forms a proportion λ_0 of the cycle.

Deduce in terms of suitably defined coefficients a_{ij} $(i = 0, 1, 2, \ldots, m\ ;\ j = 1, 2, \ldots, n)$ how the Λ_j are related to the λ_i.

What is the value of the sum of all the λ_i and why is this so?

The total lost time per cycle is L, the cycle time must not exceed c_0 and the effective green time for stage i must be at least g_{iM}, where

$$c_0 - L - \sum_{i=1}^{m} g_{iM} = mb \quad \text{and} \quad b > 0\,.$$

Show that λ^+ with $\lambda_0^+ = L/c_0$ and $\lambda_i^+ = (g_{iM} + b)/c_0$ $(i = 1, 2, \ldots, m)$ are a set of timings that satisfy all these conditions.

In stream j the arrival rate is μq_j (where μ is the same for all j) and the saturation flow is s_j. Express as a linear inequality in μ and the λ_i the requirement that the degree of saturation should not exceed p_j.

Hence show that the arrival rates μq_j can be accommodated with the timings λ^+ provided that

$$\mu \le \min_j \{\, p_j\, s_j\, [a_{0j} L + \sum_{i=1}^{m} a_{ij}(g_{iM} + b)\,]\, / q_j\, c_0 \}\,.$$

Why is this in general not the largest μ for which signal timings can be found that satisfy all the stated constraints?

2 You may assume that for any I x J matrix $\boldsymbol{\tau} = (\tau_{ij})$ whose elements are strictly positive and any strictly positive a_i $(i = 1, 2, \ldots, I)$ and b_j $(j = 1, 2, \ldots, J)$, there is one and only one matrix $\mathbf{t} = (t_{ij})$ such that $t_{ij} = r_i s_j \tau_{ij}$ for some r_i and s_j and all i and j, and the row and column sums are a_i and b_j respectively.

Show that if $\mathbf{s} = (s_{ij})$ corresponds to $\boldsymbol{\sigma} = (\sigma_{ij})$ in the same way as $\mathbf{t}$ does to $\boldsymbol{\tau}$ with the same a_i and b_j (but in general different r_i and s_j), then $\mathbf{s} = \mathbf{t}$ if and only if for some ξ_i and η_j, $\sigma_{ij} = \xi_i \eta_j \tau_{ij}$ for all i and j.

When $\tau_{ij} = \exp(-\alpha c_{ij})$ for given c_{ij} and for all i and j, the corresponding matrix $\mathbf{t}$ is denoted by $\mathbf{t}^*(\alpha)$. Show that if, for all i and a certain subset of the j, c_{ij} is increased by the same amount c_0 but the a_i and b_j are unchanged, the matrix $\mathbf{t}^*(\alpha)$ will be unchanged. (*Result* (*))

Suppose that the distribution of trips by car in Greater London were represented by the gravity model with exponential cost function. Discuss whether *Result* (*) would imply that congestion charging should have no effect on the number of journeys made by car to destinations within the charging zone.

3(a) In a stream of traffic controlled by a signal, there is no traffic queueing at time $t = 0$, the interval $(0, r)$ is effective red time and all time $t > r$ is effective green. The arrival rate and saturation flow are q and s vehicles/unit time respectively, and the amount of traffic arriving in any interval $(0, t)$ is the random variable Q_t. The random variable z $(z \geq r)$ is the time at which the queue that builds up during $(0, r)$ clears, and D_z is the delay incurred during $(0, z)$ in units of vehicle-time.

Write D_z in terms of integrals with respect to t of the two functions Q_t and $(t - r)s$.

Show that the expectation of D_z exceeds $qr^2s/2(s - q)$.

3(b) Explain what is meant by a *shock wave* in traffic, and establish an expression for the speed at which one travels.

The flow of a stream of traffic is interrupted between times $t = 0$ and $t = r$ by the effective red period of a traffic signal. At all times, the traffic approaches the signal freely at rate q and speed v, and after time $t = r$ the signal remains green indefinitely. Show that the trajectory of x_b, the back of the queue of stationary traffic, initially satisfies $x_b = \dfrac{-qlvt}{(ql - v)}$ where l is the effective length of a queued vehicle.

Using the variable s, the saturation flow, derive an expression for the time at which traffic conditions at the stop-line return to normal.

Explain why the disturbance to the traffic flow caused by the traffic signals extends upstream of position $x_r = \dfrac{-qlvr}{(ql - v)}$.

4 Wardrop's first principle of route choice is that "the journey times on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route".

Discuss the main limitations of this principle in modelling road traffic assignment.

Identify the relationship between path costs and path flows that is satisfied by the path flows $\mathbf{t}^*$ that solve the optimisation problem:

$$\underset{\mathbf{t}}{\text{Minimise}} \sum_{a \in L} \int_{v=0}^{v_a} c_a(v)\,dv + \frac{1}{\theta} \sum_{od} \sum_{p \in P_{od}} t_p \left(\log_e t_p - 1\right)$$

subject to the constraints

$$\sum_{p \in P_{od}} t_p = T_{od} \quad \forall od,$$

where

$$v_a = \sum_{od} \sum_{p \in P_{od}} t_p\, \delta_p^a \quad \forall a \in L.$$

$$\delta_p^a = \begin{cases} 1 & \text{if link } a \text{ is on path } p \\ 0 & \text{otherwise,} \end{cases}$$

$c_a(v)$ is the cost of using link a when the flow on that link is v,

θ is a positive parameter,

L is the set of links in the network,

T_{od} is the demand for travel from o to d per unit time,

t_p is the flow on path p,

P_{od} is a set of reasonable paths from o to d, and

v_a is the flow on link a.

Identify ways in which this assignment addresses limitations of Wardrop's first principle.

5 Customers arrive at a queue according to a Poisson process with mean rate q and have exponentially distributed service times with mean Q^{-1}. Show that if $x = q/Q < 1$ the equilibrium probability P_n that there are n customers in the queue is given by

$$P_n = (1-x)\,x^n \quad (n \geq 0).$$

Hence or otherwise show that $\mathrm{E}(N)$, the mean equilibrium number of customers in the queue, is given by

$$\mathrm{E}(N) = \frac{x}{(1-x)}$$

and establish an expression for the probability $\mathrm{P}(N > 0)$ that the queue is non-empty. Discuss the behaviour of this queueing system as the arrival rate q varies in $[0, Q)$.