UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C353: Theory Of Traffic Flow I

COURSE CODE	:	**MATHC353**
UNIT VALUE	:	**0.50**
DATE	:	**07-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0917-3-40

TURN OVER

All questions may be answered, but only marks obtained on the best ***four*** *questions will count. The use of an electronic calculator is* ***not*** *permitted in this examination.*

1 With a conventional notation, the average delay d to a vehicle in a stream of traffic controlled by a fixed-time signal is approximated by

$$d = 0.9\{c(1-\Lambda)^2/2(1-y) + x^2/2q(1-x)\}. \qquad (*)$$

State what is denoted by each quantity that appears in this expression and write down an inequality that is a necessary condition for the approximation to be valid.

At a signal-controlled road junction, there are n traffic streams and quantities relating to stream j are denoted by the relevant symbols from equation (*) with subscript j. The total lost time in the signal cycle is L, $\lambda_0 = L/c$ and equation (*) is valid for every stream.

Show that the average delay per unit time at the junction is approximated by $0.9D$, where

$$D = \Sigma_j \{f_j(\Lambda_j)/\lambda_0 + g_j(\Lambda_j)\},$$

where $f_j(\Lambda) = Lq_j(1-\Lambda)^2/2(1-y_j)$ and $g_j(\Lambda) = y_j^2/2\Lambda(\Lambda - y_j)$.

Regarding D as a function of the $n+1$ variables $\lambda_0, \Lambda_1, \Lambda_2, \ldots, \Lambda_n$, express in terms of λ_0, the f_j and g_j and their derivatives w.r.t. Λ (which you may write simply as f_j', g_j', f_j'' and g_j'')

(a) $\partial^2 D/\partial\lambda_0^2$
(b) $\partial^2 D/\partial\lambda_0 \partial\Lambda_j$
(c) $\partial^2 D/\partial\Lambda_j \partial\Lambda_k$

Outline briefly without detailed algebra how your results could be used to show that for all suitable signal timings the $(n+1)\times(n+1)$ matrix of these second derivatives is positive definite.

2(a) A signal-controlled road junction has two traffic streams, one having green in stage 1 and the other in stage 2 of a 2-stage cycle. The proportions of the cycle that are effectively green for stages 1 and 2 are λ_1 and λ_2 respectively.

The maximum cycle time is 80 seconds and the total lost time per cycle and the minimum green times for the two stages are each 8 seconds.

Draw in the (λ_1, λ_2) plane the triangle formed by all the points representing signal timings that satisfy the above conditions.

Draw in the same diagram the line corresponding to a cycle time of 40 seconds and let the segment of this line lying in the triangle be AB, where A corresponds to the larger of the values of λ_2 at these two points.

The amounts of traffic in the two streams are such that their degrees of saturation are unity when $\lambda_1 = 0.15$ and $\lambda_2 = 0.30$ respectively.

Without calculating any values of delay, sketch a graph of average delay per unit time at the junction as a function of distance of the point (λ_1, λ_2) along the line AB in each direction from an origin at A. You may assume that delay is minimised when $\lambda_1 \approx 0.3$.

(b) Customers arrive at a queue according to a Poisson process with mean rate q, and have independent and identical (constant) service times μ. Show that if $x = \mu q < 1$, then the mean delay d incurred by customers in this queue is given by

$$d = \frac{\mu}{2}\left(\frac{x}{1-x}\right).$$

Explain why as it stands, this expression is unsuitable for use to estimate delays at a traffic signal, and describe how it can be used within one that is more suitable.

3 Wardrop's first principle of route choice is that "the journey times on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route".

In a network in which traffic respects first-in first-out (FIFO) discipline, the demand for travel varies over time. Show that, in order for traffic flows in this network to satisfy Wardrop's first principle, the proportion $\mu_p(s)$ of demand that is assigned to route p at time s satisfies

$$\mu_p(s) = \frac{g_p[\tau(s)]}{\sum_{q \in P} g_q[\tau(s)]} \quad \forall p \in P$$

where $\tau(s)$ is the time of arrival of traffic that departs at time s,
$g_p(t)$ is the outflow from route p at time t, and
P is the set of routes available for that journey.

Discuss the use in the right-hand side of this expression of route outflows at time $\tau(s)$ to calculate assignment proportions at time $s < \tau(s)$.

4 Explain clearly the physical interpretation and derive from first principles a formula for the speed of each of a *shock wave* and a *traffic wave*. Show that at each density, vehicles travel faster than do waves and establish an expression for the rate at which traffic passes a wave of density k.

A traffic signal is located at position $x = 0$ on a road, where x increases in the direction of travel and traffic generally moves according to the speed-density relationship

$$v = v_0\left(1 - \frac{k}{k_j}\right) \quad (0 \le k \le k_j)$$

where v_0 is the free-flow speed and k_j is the jam density. Before time $t = -r$ the signal is green and the traffic flow is uniform everywhere at rate q_a. At that time, the traffic signal changes to red so that a queue of density $k = k_j$ starts to form in the region $x < 0$. At time $t = 0$, the signal changes to green and traffic in the queue starts to move according to the speed-density relationship. Show that the maximum upstream extent of the queue occurs when the density k_f at the back of the queue is given by

$$k_f = \frac{k_j}{2}\left(1 + \sqrt{1 - \frac{4\,q_a}{v_0\,k_j}}\right).$$

Hence or otherwise show that the position x_m and time of occurrence t_m of the maximum upstream position of the queue are given by

$$(x_m, t_m) = \left(-v_0\sqrt{1 - \frac{4\,q_a}{v_0\,k_j}}\,,\,1\right)\frac{q_a\,r}{(v_0\,k_j - 4\,q_a)}.$$

5 Given an $I \times J$ matrix $\boldsymbol{\tau} = (\tau_{ij})$ whose elements are strictly positive and sets of desired strictly positive row and column sums a_i $(i = 1, 2, \ldots, I)$ and b_j $(j = 1, 2, \ldots, J)$, define the two kinds of iteration in the Furness iterative procedure whose limit is the unique matrix $\mathbf{t}^* = (t_{ij}^*)$ with the desired row and column sums and such that

$$t_{ij}^* = r_i s_j \tau_{ij}$$

for some r_i and s_j and all i and j.

A second matrix $\boldsymbol{\sigma} = (\sigma_{ij})$ has limit $\mathbf{s}^*$ under the Furness procedure. Show that $\mathbf{s}^* = \mathbf{t}^*$ if and only if for some ξ_i and η_j, $\sigma_{ij} = \xi_i \eta_j \tau_{ij}$ for all i and j.

Suppose that, for all i and j, $\tau_{ij} = \exp(-\alpha c_{ij})$, where α is a parameter and c_{ij} is the cost of travel from origin zone i to destination zone j in a city, and that t_{ij}^* is an estimate of the number of journeys from i to j in a certain period. In a city in which $I = 2$ and $J = 3$ and α, the a_i and the b_j are given, which of the following four matrices (c_{ij}) would lead to a different $\mathbf{t}^*$ from the other three and why?

$$\begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \qquad \begin{pmatrix} 4 & 5 & 6 \\ 6 & 7 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 & 6 \\ 6 & 7 & 9 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 6 \\ 7 & 8 & 8 \end{pmatrix}$$