UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M323: Semisimple Algebras and Group Representations

COURSE CODE	:	**MATHM323**
UNIT VALUE	:	**0.50**
DATE	:	**27-APR-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1056-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Define the *trace* $Tr(A)$ of A and show that if B is also an $n \times n$ matrix over $\mathbb{F}$ then $Tr(BA) = Tr(AB)$.

   If G is a finite group explain what is meant by

   (i) a *representation* of G over $\mathbb{F}$, and

   (ii) the *character* χ_ρ of a representation ρ of G.

   If ρ, σ are n-dimensional representations of G such that $\chi_\rho \neq \chi_\sigma$ prove that ρ is not conjugate to σ.

   Hence show that the representations ρ, σ of the cyclic group $C_4 = \langle x \mid x^4 = 1 \rangle$ given below are not conjugate provided $1 + 1 \neq 0$ in $\mathbb{F}$;

$$\rho(x) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \quad \sigma(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

2. State Schur's Lemma.

   Let G be a finite group and let $\mathbb{F}$ be a field in which $|G|$ is invertible. If M is a finitely generated module over $\mathbb{F}[G]$ and $N \subset M$ is a submodule, prove that $M \cong N \oplus (M/N)$.

   Explain briefly the main steps in showing that $\mathbb{F}[G]$ is a direct product

$$\mathbb{F}[G] \;\cong\; M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$$

   where $D_1, \ldots, D_m$ are finite dimensional division algebras over $\mathbb{F}$.

   In the case where $\mathbb{F} = \mathbb{C}$, state, without proof, an interpretation of m in terms of G, and find m and $d_1, \ldots, d_m$ when G is the nonabelian group of order 21 defined by the presentation

$$G(21) = \langle x, y \mid x^7 = y^3 = 1,\; yx = x^2 y \rangle .$$

3. Let $\rho_1, \ldots, \rho_m$ be the distinct simple representations of the finite group G over $\mathbb{C}$ where $\rho_i : G \to GL_{d_i}(\mathbb{C})$ and let ϵ_i be the central idempotent of $\mathbb{C}[G]$ associated with ρ_i. State and prove a formula which expresses ϵ_i in terms of the character χ_i of ρ_i.

Hence show that $\sum_{g \in G} \chi_i(g^{-1})\chi_j(g) = |G| \, \delta_{ij}$.

Find the central idempotent ϵ_i explicitly when G is the alternating group A_4 of order 12

$$G = A_4 = \langle s, t, x \mid s^2 = t^2 = (st)^2 = 1, xsx^{-1} = t, xtx^{-1} = st \rangle$$

and $\rho_i : A_4 \to GL_3(\mathbb{C})$ is the simple representation defined on generators by

$$\rho_i(s) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; ; \; \rho_i(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; ; \; \rho_i(x) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

4. If $\mathcal{A}$, $\mathcal{B}$ are algebras over a field $\mathbb{F}$ describe briefly how the tensor product $\mathcal{A} \otimes \mathcal{B}$ may also be regarded as an algebra.

Explain how a group ring of the form $\mathbb{F}[G \times H]$ may be described as a tensor product algebra.

Describe briefly a calculation which shows that $\mathrm{M}_m(\mathbb{F}) \otimes \mathrm{M}_n(\mathbb{F}) \cong \mathrm{M}_{mn}(\mathbb{F})$. (You may assume without proof the standard rules for manipulating the tensor symbol $\otimes$.)

A certain group Γ of order 20 has the Wedderburn decomposition

$$\mathbb{C}[\Gamma] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathrm{M}_4(\mathbb{C}).$$

Describe the Wedderburn decomposition of $\Gamma \times \Gamma$; hence find

(i) the number of distinct simple 4-dimensional representations of $\Gamma \times \Gamma$;

(ii) the largest dimension of any simple representation of $\Gamma \times \Gamma \times \Gamma$.

5. A finite group G is expressed in the form $G = KQ$ where K, Q are subgroups of G such that $K \lhd G$ and $K \cap Q = \{1\}$, and denote by $\varphi : Q \to \mathrm{Aut}(K)$ the homomorphism $\varphi(q)(k) = qkq^{-1}$ for $q \in Q$ and $k \in K$. If $\rho : K \to \mathrm{GL}_n(\mathbb{C})$ is a representation explain how to construct the induced representation

$$\mathrm{Ind}_K^G(\rho) : G \longrightarrow \mathrm{GL}_{n|Q|}(\mathbb{C}).$$

Let $G = \langle x, y \mid x^9 = y^2 = 1,\ yx = x^{-1}y \rangle$ be the dihedral group of order 18 ; taking $K = \langle x \rangle$ to be the normal subgroup generated by x and $Q = \langle y \rangle$ to be the subgroup generated by y, let $\rho : K \to \mathrm{GL}_1(\mathbb{C})$ be the representation given by

$$\rho(x) \;=\; \omega^2 \;; \qquad (\; \omega = \exp(\frac{2\pi i}{3}) \;).$$

Calculate the explicit form of the matrices representing x and y in the induced representation $\mathrm{Ind}_K^G(\rho)$.

Hence show that $\mathrm{Ind}_K^G(\rho)$ is simple.

END OF PAPER