UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M323: Semisimple Algebras and Group Representations

COURSE CODE : **MATHM323**

UNIT VALUE : **0.50**

DATE : **12-MAY-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0970-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let R be a ring, and let M be a right R-module. Explain what is meant by saying that M is *simple.*

   State and prove Schur's Lemma.

   Let $\varphi : M_1 \oplus \ldots \oplus M_m \to N$ be a surjective R-homomorphism where $M_1, \ldots M_m$ are simple nonzero R-modules and N is nonzero ; prove there exists a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ such that $\varphi : M_{i_1} \oplus \ldots \oplus M_{i_k} \to N$ is an isomorphism.

   State and prove a theorem which describes the structure of finitely generated modules over $M_n(D)$ where D is a division ring.

2. Let G be a finite group and let $\mathbb{F}$ be a field in which $|G|$ is invertible. If M is a finitely generated module over $\mathbb{F}[G]$ and $N \subset M$ is a submodule, prove that $M \cong N \oplus (M/N)$.

   Explain briefly the main steps in showing that $\mathbb{F}[G]$ is a direct product

$$\mathbb{F}[G] \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$$

   where $D_1, \ldots, D_m$ are finite dimensional division algebras over $\mathbb{F}$.

   In the case where $\mathbb{F} = \mathbb{C}$, state, without proof, an interpretation of m in terms of G, and find m and $d_1, \ldots, d_m$ when $G = A_4$, the group of even permutations of the set $\{1, 2, 3, 4\}$.

3. Define the *trace*, $Tr(A)$, of an $n \times n$ matrix A, and show that $Tr(AB) = Tr(BA)$.

   Define the *character* χ_ρ of a representation $\rho : G \to GL_d(\mathbb{C})$ of a finite group G,

   Explain what is meant by saying that a representation $\sigma : G \to GL_d(\mathbb{C})$ is *equivalent* to ρ, and show that, in this case, $\chi_\sigma = \chi_\rho$.

   Let Q_8 denote the quaternion group $Q_8 = \langle x, y \mid x^2 = y^2, yxy^{-1} = x^3 \rangle$.

   Describe, up to equivalence, all simple representations of Q_8 over $\mathbb{C}$, and, by writing down the character table of Q_8, explain why your description is complete.

4. Let G be a finite group, let $\rho : G \to GL_d(\mathbb{C})$ be a simple representation with character χ_ρ, and let ϵ_ρ be the central idempotent of $\mathbb{C}[G]$ associated with ρ. State and prove a formula which expresses ϵ_ρ in terms of χ_ρ.

   Let $\chi_1, \ldots, \chi_m$ be the distinct simple characters of G over $\mathbb{C}$. State the First and Second Orthogonality relations for $\chi_1, \ldots, \chi_m$.

   Show, furthermore, how the first relation follows from the idempotent formula, and explain briefly how the second relation follows from the first.

   Find ϵ_ρ explicitly when G is the binary dihedral group

$$G = D_6^* = \langle x, y \mid x^3 = y^4 = 1, yxy^{-1} = x^2 \rangle$$

   and $\rho : D_6^* \to GL_2(\mathbb{C})$ is the representation

$$\rho(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \; ; \; \rho(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

5. Let H be a subgroup of the finite group G, and let $\mathbb{F}$ denote a field. Describe the 'extension of scalars' construction $\mathcal{E}_H^G : \{\mathbb{F}[H]\text{−modules}\} \to \{\mathbb{F}[G]\text{−modules}\}$, and explain briefly its relation to the classical construction of the representation $\mathrm{Ind}_H^G(\rho)$ of G induced from a representation $\rho : H \to GL_k(\mathbb{F})$.

   Let G be the dihedral group of order 8 :

$$G = \langle x, y \mid x^4 = y^2 = 1, \; yx = x^3 y \rangle,$$

   and let K be the subgroup $K = \langle x \rangle$ generated by x. If ρ_K is the representation

$$\rho_K : K \to GL_1(\mathbb{C}) \; ; \; \rho_K(x) = \sqrt{-1}$$

   show that $\mathrm{Ind}_K^G(\rho_K)$ is the unique irreducible two-dimensional complex representation of G.