UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M323: Semisimple Algebras and Group Representations

COURSE CODE	:	**MATHM323**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0957-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Explain what is meant by a *semisimple* module.

   Let $\mathcal{A}$ be a finite dimensional associative algebra over a field $\mathbb{F}$. If $\mathcal{A}$ is semisimple as a module over itself, explain briefly the main steps in showing that $\mathcal{A}$ is a direct product

$$\mathcal{A} \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$$

   where $D_1, \ldots, D_m$ are finite dimensional division algebras over $\mathbb{F}$.

   State and prove Maschke's Theorem.

   If G is a finite group and

$$\mathbb{C}[G] \cong M_{d_1}(\mathbb{C}) \times \ldots \times M_{d_m}(\mathbb{C})$$

   state, with proof, an interpretation of m in terms of G.

2. Define the trace, $\mathrm{Tr}(A)$, of an $n \times n$ matrix A and show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

   If $\rho : G \to GL_d(\mathbb{C})$ is a representation of the finite group G, define the *character* $\mathcal{X}_\rho$, of ρ and prove that $\mathcal{X}_\rho$ is constant on conjugacy classes.

   Consider the group

$$D_8 = \langle x, y | x^4 = y^2 = 1, yxy^{-1} = x^3 \rangle .$$

   (i) Find the Wedderburn decomposition of D_8.

   (ii) Describe, up to isomorphism, all simple representations of D_8 over $\mathbb{C}$.

   (iii) Write down the character table of D_8.

3. Let $\mathcal{X}, \ldots, \mathcal{X}_m$ be the distinct simple characters over $\mathbb{C}$ of a finite group.

State the First and Second Orthogonality relations for $\mathcal{X}_1, \ldots, \mathcal{X}_m$ and show how the second is derived from the first.

Let $\rho : G \to GL_d(\mathbb{C})$ be a simple representation of G with character $\mathcal{X}_\rho$. Give a formula which expresses the primitive central idempotent $\in_\rho$, associated with ρ, in terms of $\mathcal{X}_\rho$.

Find $\in_\rho$ explicitly when

$$G = A_4 = \langle x, y, \sigma \,|\, x^2 = y^2 = (xy)^2 = \sigma^3 = 1\,,\, \sigma x \sigma^{-1} = xy\,,\, \sigma y \sigma^{-1} = x \rangle$$

and $\rho : A_4 \to GL_3(\mathbb{C})$ is the representation

$$\rho(x) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

4. State Burnside's Theorem on groups of order $p^a q^b$ explaining all essential terms in the statement.

Let G be a finite group and let $\mathcal{X}$ be the character of the representation

$$\rho : G \to GL_d(\mathbb{C}).$$

If $g \in G$ show that

(i) $|\mathcal{X}(g)| \leqslant d$ and

(ii) $|\mathcal{X}(g) = d \Leftrightarrow \rho(g)$ belongs to the centre of $GL_d(C)$.

[You may assume without proof that $\rho(g)$ is conjugate to a diagonal matrix.]

Explain how (ii) is used to prove a nonsimplicity criterion for G.

Show that S_4, the group of permutations on 4 letters, is soluble by describing a composition series for S_4 explicitly.

5. Let V, W be vector spaces over a field $\mathbb{F}$. Explain, with reference to a suitable universal property, what is meant by the tensor product $V \otimes_{\mathbb{F}} W$ when $\dim_{\mathbb{F}}(V)$ is finite.

If V, W are also algebras over $\mathbb{F}$, describe the corresponding algebra structure on $V \otimes_{\mathbb{F}} W$.

If G_1, G_2 are finite groups, describe the algebra structure on $\mathbb{F}[G_1 \times G_2]$ in terms of the tensor product construction.

Suppose that

$$\begin{aligned}\mathbb{R}[G_1] &\cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H} \\ \mathbb{R}[G_2] &\cong \mathbb{R} \times \mathbb{R} \times M_2(\mathbb{R})\end{aligned}$$

Find

(i) the Wedderburn decomposition of $\mathbb{R}[G_1 \times G_2]$

(ii) the Wedderburn decomposition of $\mathbb{C}[G_1 \times G_2]$

(iii) the largest degree of any $\mathbb{C}$-simple representation of $G_1 \times G_2$.