UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics 3205: Riemann Surfaces and Algebraic Curves

COURSE CODE	:	**MATH3205**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0904-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let C be the algebraic curve in $\mathbb{P}^2$ defined by the polynomial

$$F(X,Y,Z) = X^3 + Y^3 - 2X^2Z + Y^2Z + XZ^2$$

   (a) Find the singular points of C, the multiplicities of C at these points and the equations of the tangent lines at these points.

   (b) Let L_1 be the line in $\mathbb{P}^2$ defined by $X = 0$, let L_2 be the line $Y = 0$ and let D be the curve $2X^2 - Y^2 - XZ = 0$. Let P be the point $[0 : 0 : 1]$.
   Compute $I_P(C, L_1)$, $I_P(C, L_2)$ and $I_P(C, D)$.

2. (a) Let X be a pathwise connected, locally simply connected topological space. Let x_0 be a point of X. Say what is meant by

   (i) the fundamental group $\pi_1(X, x_0)$

   (ii) a covering space of X

   (iii) a covering transformation and a Galois covering

   (iv) the universal covering space of X.

   (b) Let T be a torus and P a point of T. Determine the fundamental group of the space obtained by removing the point P from T. (Justification not required)

   (c) Let X be the wedge of 2 circles. Recall that the fundamental group of X is the free group $< a, b >$ on two generators a and b. Describe the covering spaces of X corresponding to the following subgroups:

   (i) $< a^3, b^3, ab, ba >$

   (ii) $< a^2, b^2, aba^{-1}, bab^{-1} >$

3. (a) Let X be a compact Riemann surface of genus g_X. Explain what is meant by the Euler-Poincaré characteristic $\chi(X)$ and give the expression for $\chi(X)$ in terms of g_X.

   Let Y be a compact Riemann surfaces of genus g_Y and let $\pi: X \longrightarrow Y$ be a holomorphic map. If y is a point of X, explain what is meant by

   (i) the ramification index of π at x and by saying that π is ramified at x.

   (ii) the degree of π.

   State the Riemann-Hurwitz formula.

   (b) Suppose that $\pi: X \longrightarrow Y$ has degree two. Show that the number of ramification points is even.

   (c) If, in addition $Y = \mathbb{P}^1$ and n denotes the number of ramification points, express g_X in terms of $m = \frac{n}{2}$.

   For each $m \geqslant 1$, give an example of a double covering $\pi: X \longrightarrow \mathbb{P}^1$ ramified at $2m$ points.

4. (a) Let C and D be two curves in $\mathbb{P}^2$ of degrees n and m respectively. State the Bezout's theorem for C and D.

   (b) Let C be a projective curve of degree d and suppose that C has m singular points lying on a line L.

   (i) Show that if $2m > d$, then L is contained in C.

   (ii) Show that if C is irreducible, then $2m \leqslant d$.

   (c) Let C_n be the algebraic curve in $\mathbb{P}^2$ given by the equation $X^n + Y^n + Z^n = 0$ where $n \geqslant 3$. Let $f: [x : y : z] \mapsto [x : z]$ be a holomorphic map $C_n \longrightarrow \mathbb{P}^1$. Determine the ramification points of f and their indices. Find the genus of C_n.

5. (a) Let X be a compact Riemann surface of genus g and D a divisor on X.

   (i) Define the vector space $\mathcal{L}(D)$.

   (ii) Let $l(D)$ be the dimension of $\mathcal{L}(D)$. Show that if $\deg D < 0$ then $l(D) = 0$.

   (iii) Show that if D and D' are linearly equivalent, then the vector spaces $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic.

   (iv) Explain what is meant by a canonical divisor and prove that all canonical divisors are linearly equivalent.

   (v) State the Riemann-Roch theorem for X.

   (b) Let X be a compact Riemann surface of genus g and P a point of X. Show that there exists a meromorphic function on X having P as a unique pole.

   (c) Let $P_1, \ldots, P_r$ be r points on a compact Riemann surface X of genus g. Show that there exists a meromorphic function having poles at $P_1, \ldots, P_r$ and holomorphic elsewhere.