UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C325: Real Fluids

COURSE CODE	:	**MATHC325**
UNIT VALUE	:	**0.50**
DATE	:	**17-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1006-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

Unless stated otherwise, you may assume that external (body) forces are absent.

1. Write down the constitutive relation for an isotropic Newtonian fluid in terms of the stress tensor, σ_{ij}, and the rate of strain tensor, e_{ij}.

   Starting from the Cauchy equations of motion,

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

   derive the Navier-Stokes momentum equations for an incompressible Newtonian fluid in the form

$$\rho \left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right] = -\nabla p + \mu \nabla^2 \mathbf{U}$$

   in the usual notation.

   The rate of strain tensor in a *steady* two-dimensional flow is given by the relations $e_{11} = 0$, $e_{12} = e_{21} = b$, $e_{22} = 0$, where b is a constant. Determine the velocity and pressure distribution in the flow and give a qualitative interpretation of the flow field in terms of the streamfunction. Discuss the uniqueness or otherwise of the solution obtained.

2. Incompressible fluid with density ρ and kinematic viscosity ν is contained in a channel between two parallel plates at $y = 0$ and $y = d$ in the (x, y)-plane. The flow is driven by an oscillatory pressure gradient, $\partial p / \partial x = -\rho [G_1 + G_2 \cos(\omega t)]$ with constant G_1, G_2 and ω. Show that a unidirectional flow is possible with the velocity along the channel of the form

$$u(y, t) = u_1(y) + \mathrm{Re}[\mathrm{e}^{i\omega t} u_2(y)]$$

   and determine the form of the functions $u_1(y)$ and $u_2(y)$.

   Discuss briefly the limiting forms of the velocity profile in the flow for large and small values of the frequency ω.

3. Incompressible viscous fluid is contained between coaxial cylinders of radii a and b, $a < b$, rotating about their axis with angular velocities Ω_a and Ω_b respectively. The fluid is injected through the surface of the inner cylinder with radial velocity $V > 0$ and absorbed radially through the surface of the outer cylinder. Assuming that the resulting flow is steady and axi-symmetric,

   (i) find the radial velocity of absorption on the outer cylinder;

   (ii) find the azimuthal velocity u_θ in the flow;

   (iii) comment briefly on the properties of the flow in the limit of small kinematic viscosity, $\nu \to 0$.

   You may assume the Navier-Stokes equations in cylindrical polar coordinates (r, θ, z) in the form

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right);$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right);$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right);$$

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

4. State the assumptions of the lubrication approximation for a two-dimensional incompressible flow in a narrow gap between two solid boundaries.

   Incompressible fluid with coefficient of viscosity μ is in a narrow two-dimensional channel between a fixed solid plate at $y = 0$ and a moving boundary whose position is given by $y = h(\xi)$ where $\xi = x - c_0 t$ with constant c_0. Each point of the boundary moves in the x-direction with constant speed c_1. Show that the no-slip conditions on the moving boundary can be written as

$$u = c_1, v = (c_1 - c_0)\, h'(\xi)$$

   where u, v are the velocity components in the x, y directions, respectively.

   Show that under the assumptions of the lubrication approximation the pressure in the channel, $p(\xi)$, is governed by the equation

$$\left(p'(\xi)\, h^3(\xi)\right)' = 12\mu \left(\frac{c_1}{2} - c_0\right) h'(\xi)\,.$$

   Obtain an explicit solution for the pressure for the case $h = h_0 + \alpha\xi$, with constant h_0, α, and $0 \leqslant \xi \leqslant L$, assuming also that the pressure is a given constant, $p = p_0$, at the end points at $\xi = 0$ and $\xi = L$.

5. Show that, for slow steady flow of incompressible fluid, the vector potential $\mathbf{A}$ satisfies the equation

$$\nabla^2\left(\nabla^2 \mathbf{A}\right) = \mathbf{0}.$$

   The fluid is injected slowly in the radial direction into a circle of radius a through some sections of the circumference and removed from the circle through the other sections. The radial velocity on the boundary of the circle is $u_r = u_0 \cos\theta$, where u_0 is a constant and θ is the polar angle. By changing to cylindrical polar coordinates, r, θ, show that the streamfunction ψ for the flow inside the circle can be found in a separable form, $\psi = f(r)\, g(\theta)$ and determine the functions f and g.

   You may assume the formulae

$$\mathrm{div}\mathbf{U} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta},$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$