UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C325: Real Fluids

COURSE CODE	:	**MATHC325**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0921-3-50

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Let $V(t)$ be a material volume of an incompressible fluid of density ρ with the surface $S(t)$. Write down a relationship between the components of the stress vector, Σ, and the stress tensor, σ_{ij}, at some point on the surface $S(t)$. Show that in the absence of body forces Newton's momentum equation,

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{U} dV = \int_{S(t)} \Sigma dS,$$

leads to the Cauchy equations of motion,

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

in the usual notation.

Incompressible fluid with density ρ and viscosity μ flows in a straight duct whose axis coincides with the x-axis of the Cartesian coordinates (x, y, z). Show that, regardless of the shape of the duct in the cross-section, a unidirectional flow along the duct is governed by the equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where $p = p(x, t)$ is the pressure and $u = u(y, z, t)$ is the axial velocity component. Hence, or otherwise, derive an exact solution for the steady Poiseuille flow in the gap between two co-axial cylinders of radii a and $b, a < b$, driven by a constant pressure gradient, $\partial p/\partial x = -G$. Discuss briefly the limit $a \to 0$ in the cases when r is of $O(1)$ and when $R = r/a$ is of $O(1)$.

You may assume that

$$\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

in cylindrical polar coordinates.

2. Incompressible fluid with kinematic viscosity ν is in a channel formed by two parallel plates at $y = \pm d$, $-\infty < x < \infty$. The fluid is initially at rest. At time $t \geq 0$ the walls of the channel move along the x-axis with the same constant speed u_0 but in opposite directions so that $u = \pm u_0$ at $y = \pm d$. Show that a uni-directional flow is possible with the velocity component along the channel of the form

$$u(y,t) = \sum_{n=0}^{\infty} e^{\lambda_n t} f_n(y)$$

with $\lambda_0 = 0$. Determine the time-independent part of the velocity, $f_0(y)$, the constants λ_n, and the form of $f_1(y)$ in terms of u_0, d and ν.

Discuss briefly the shape of the velocity distribution in the channel at small and large values of t.

3. Write down the Navier-Stokes equations in component form for a two-dimensional unsteady flow in the (x, y)-plane. Hence, or otherwise, show that elimination of the pressure gradient terms gives the following vorticity equation,

$$\frac{\partial \omega}{\partial t} + (\mathbf{U} \cdot \nabla)\,\omega = \nu \nabla^2 \omega,$$

where $\mathbf{U} = (u, v)$ is the velocity vector and $\omega = \partial u/\partial y - \partial v/\partial x$ is the vorticity in the flow.

Suppose that the flow is periodic in the x- and y-directions with period a and the flow speed is small so that the convective derivative in the vorticity equation can be omitted. Show that the average over the period of the vorticity squared defined by

$$\omega_m^2(t) = \int_0^a \int_0^a \omega^2(x, y, t)\, dx dy$$

is a monotonically decaying function of time.

4. Write down the equations of lubrication approximation for a two-dimensional flow in a thin film of a heavy fluid on a flat horizontal surface at $y = 0$.

   The upper surface of the film, at $y = h(x,t)$, is in contact with air. Assuming that the air pressure is constant and the shear at the film surface, $\mu \partial u/\partial y = \tau(x,t)$ is a given function, show that the evolution of the film surface is governed by the equation

$$\frac{\partial h}{\partial t} + \frac{1}{2\mu}\frac{\partial}{\partial x}\left(\tau h^2\right) - \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) = 0,$$

   where g is the acceleration due to gravity and ρ, μ are the density and viscosity of the fluid, respectively.

   Let $\tau = \tau_0$ be a constant. Verify that a flow is possible with a constant film thickness, $h = h_0$. Next consider small perturbations to the constant-thickness solution in the form of travelling waves,

$$h(x,t) = h_0 + \varepsilon \exp\left[i(kx - \omega t)\right] + O\left(\varepsilon^2\right),$$

   where the wave amplitude, ε, and the terms of order ε^2 and higher can be neglected. Derive a dispersion relation for the frequency ω in terms of the wavenumber, k. Hence or otherwise deduce that the flow is stable.

5. Derive the Stokes equation,

$$\nabla^2\left(\nabla^2\psi\right) = 0,$$

   for the streamfunction in a two-dimensional, slow steady flow of incompressible fluid..

   A corner of half-angle α has its vertex at the origin of the polar coordinates ρ, θ and solid walls at $\theta = \pm\alpha$. Show that the streamfunction for a slow flow in the corner can be found in the form $\psi = r^\lambda f(\theta)$ provided the constant λ satisfies the equation

$$\lambda \tan(\lambda\alpha) = (2 - \lambda)\tan\left[(2 - \lambda)\alpha\right].$$

   It can be assumed that the streamfunction is an even function of θ.

   In the case $\alpha = \pi$, find the smallest admissible positive value of λ.

   You may assume that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$