UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C325: Real Fluids

COURSE CODE	:	**MATHC325**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0908-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

You may assume that external (body) forces are absent.

1. Write down the constitutive relation for an isotropic Newtonian fluid in terms of the stress tensor, σ_{ij}, and the rate of strain tensor, e_{ij}.

   Starting from the Cauchy equations of motion,

   $$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

   derive the Navier-Stokes momentum equations for an incompressible Newtonian fluid in the form

   $$\rho \left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\, \mathbf{U} \right] = -\nabla p + \mu \nabla^2 \mathbf{U}$$

   in standard notation.

   The rate of strain tensor in a *steady* two-dimensional flow is given by the relations $e_{xx} = a$, $e_{xy} = e_{yx} = 0, e_{yy} = -a$, where a is a constant and the usual notation is assumed for the coordinates and velocity components in two dimensions, namely $x_1 = x$, $x_2 = y$, $u_1 = u$, $u_2 = v$. Show that a flow of incompressible viscous fluid is possible with such a rate of strain tensor and determine the velocity and pressure distribution in the flow. Give a qualitative interpretation of the flow obtained.

2. Incompressible viscous fluid with kinematic viscosity ν fills a two-dimensional channel formed between two parallel plates, $0 \le y \le d, -\infty < x < \infty$. The lower wall of the channel, at $y = 0$, oscillates parallel to the x-axis with velocity $u = u_0 \cos(\omega t)$, where u_0 and ω are constant. Show that a uni-directional flow is possible with the velocity component in the x-direction of the form,

   $$u = u_0 \,\mathrm{Re} \left\{ e^{i\omega t} \frac{\sinh\left[\lambda\,(d-y)\right]}{\sinh(\lambda d)} \right\},$$

   with $\lambda = (1+i)\,(\omega/2\nu)^{1/2}$. Hence, or otherwise, obtain the velocity distribution in the channel in two limiting cases, (i) $d \to \infty$, and (ii) $d \to 0$, and give a physical interpretation of the limit solutions.

3. Show that the pressure gradient terms in the Navier-Stokes equations for a two-dimensional flow of an incompressible viscous fluid with kinematic viscosity ν can be eliminated to obtain the following equation for the vorticity ω,

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega,$$

where $\omega = \partial u/\partial y - \partial v/\partial x$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and u and v are the velocity components in the x and y directions, respectively.

Hence, or otherwise, verify that the streamfunction in the form $\psi = f(t)\cos(\alpha x)\cos(\beta y)$ provides an exact solution of the Navier-Stokes equations if $f(t) = C\exp(-\lambda t)$ with an arbitrary constant C and the constant parameter λ determined uniquely in terms of α and β.

Sketch the streamlines in the flow at some fixed moment in time.

4. State the assumptions of the lubrication approximation for a steady, two-dimensional incompressible flow in a narrow gap between two solid boundaries.

Incompressible viscous fluid is contained in a narrow two-dimensional channel formed by a solid wall at $y = 0$ and an elastic surface at $y = f(x,t)$, with $-\infty < x < \infty$. Assuming that the lubrication approximation holds, show that deformations of the elastic wall are governed by an equation of the form

$$f_t = \frac{1}{6\mu}\left(f^3 p_x\right)_x$$

where μ is the viscosity coefficient and p is the fluid pressure in the channel.

The pressure outside the channel is maintained constant and equal to p_0. When the pressure in the channel matches the outer pressure, $p = p_0$, the fluid is at rest and the elastic surface remains flat, $f = f_0$. Variations of the fluid pressure in the channel lead to deformations of the elastic wall in accordance with the wall equation,

$$p_0 - p = \alpha(f - f_0) + \beta f_x,$$

with constant (positive or negative) parameters α and β. Consider small wave-like fluctuations in the shape of the elastic surface and show that (i) the flow in the channel can sustain wave motion of small amplitude if $\alpha = 0$, and (ii) the wave motion will be either amplified or damped, depending on the sign of α, in the case $\alpha \neq 0$.

5. Show that the streamfunction, $\psi(x, y)$, for a **slow** two-dimensional and steady flow of an incompressible viscous fluid is governed by the Stokes equation,

$$\nabla^2\left(\nabla^2\psi\right) = 0.$$

Verify that the streamfunction for a Stokes flow in a corner of angle 2α can be found in the form $\psi = r^\lambda f(\theta)$ provided that ψ is an even function of θ, the flow region is specified by the conditions $-\alpha \leq \theta \leq \alpha$ in polar coordinates (r, θ), and the constant λ satisfies the equation

$$\lambda \tan(\lambda\alpha) = (\lambda - 2)\tan[(\lambda - 2)\alpha].$$

In the case $\alpha = \pi/4$, show that the equation for λ allows a countable set of solutions, $\lambda = \lambda_n$, $n = 0, \pm 1, \pm 2, \ldots$, with $\lambda_n \approx 4n$ as $n \to \pm\infty$.

You may assume the formula

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$