UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B3D: Pure Mathematics

COURSE CODE	:	**MATHB03D**
UNIT VALUE	:	**0.50**
DATE	:	**11-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C0998-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcl} 2w & + & x & + & y & - & 4z & = & 0 \\ w & + & x & + & y & + & z & = & 4 \\ 3w & + & 2x & + & 2y & - & 3z & = & C \end{array}$$

   (a) Write the system in the form of a matrix equation.

   (b) Reduce the augmented matrix of this system to echelon form. For what value of C does the system have a solution? What is the rank of the augmented matrix in this case?

   (c) Find the general solution of the system (with your value of C from part (b) above) in the form $\underline{a} + s\underline{v}_1 + t\underline{v}_2$ where $\underline{a}$, $\underline{v}_1$ and $\underline{v}_2$ are vectors and s and t are scalars.

   (d) Apply the Gram-Schmidt process to the pair of vectors $\underline{v}_1$, $\underline{v}_2$ to give two new orthonormal vectors $\hat{\underline{g}}_1$ and $\hat{\underline{g}}_2$.

2. (a) Explain what is meant by eigenvalues and eigenvectors of a square matrix $\underline{\underline{A}}$.

   (b) Find the eigenvalues and eigenvectors of the following matrix:

$$\underline{\underline{A}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

   (c) Find the general solution, $x(t)$ and $y(t)$, to the pair of ordinary differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y - 1; \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 2y + 2.$$

   (d) A particular solution to the system of (c) above has $x(0) = 7$ and $y(0) = 6$. Find the explicit form of $x(t)$ and $y(t)$ in this case.

3. (a) Find the general solution to the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = -3x^3.$$

(b) A solution $y(x)$ to the differential equation in (a) above satisfies

$$y(1) = 0; \qquad y(2) = 0.$$

Find the explicit form of $y(x)$, and calculate $y(3)$.

(c) Find the general solution to the differential equation

$$\frac{\mathrm{d}^2f}{\mathrm{d}t^2} - 4\frac{\mathrm{d}f}{\mathrm{d}t} + 5f = 0.$$

4. The Fourier series for a function $f(x)$ with period 2π is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

(a) Write down formulae for the Fourier coefficients a_n and b_n.

(b) What can be said about the coefficients a_n and b_n if $f(x)$ is
(i) odd; (ii) even?

A function $f(x)$ with period 2π is defined by

$$f(x) = x \qquad -\pi \leq x < \pi$$

(c) Sketch the graph of the function for three periods. Is it odd or even?

(d) Calculate the Fourier coefficients.

(e) Using Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)f(x)\,\mathrm{d}x = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

5. Let $f(x, y)$ be a real valued function of two variables x and y.

   (a) Explain what is meant by a critical point of the function $f(x, y)$.

   (b) Describe the possible types of *non-degenerate* critical point that can occur.

   (c) Give a condition in terms of second partial derivatives of f, under which the critical point is non-degenerate.

   Consider the function

$$f(x, y) = x^2y + xy^2 + 2xy + 2y^2.$$

   (d) Find all of the first and second order partial derivatives of f.

   (e) Find and classify the four critical points of f.

6. Consider the vector field

$$\underline{u}(x, y, z) = (xy, x + y, 3z^2 + x).$$

   (a) Calculate $\underline{\nabla} \cdot \underline{u}$, the divergence of $\underline{u}$.

   (b) Calculate $\underline{\nabla} \times \underline{u}$, the curl of $\underline{u}$.

   Now consider the scalar function

$$f(x, y) = x^2 + y^2 + \frac{y}{(x^2 + y^2)^{1/2}}.$$

   Calculate the Laplacian $\nabla^2 f$:

   (c) directly; and

   (d) by converting into plane polar coordinates and using the standard form

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

7. (a) Consider the complex conjugate operator G, which acts on a complex number $z = x + iy$:

$$G \; : \; x + iy \longrightarrow x - iy.$$

Is it a linear operator? Give your reasons.

(b) Explain what is meant by the *commutator* $[L_1, L_2]$ of two linear operators L_1 and L_2.

(c) Show that, for any three linear operators A, B and C,

$$[A \circ B, C] = A \circ [B, C] + [A, C] \circ B.$$

(d) Consider the three linear operators (acting on a two-dimensional vector)

$$L_A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad L_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad L_C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(i) Calculate $[L_B, L_C]$;

(ii) Calculate $[L_A, L_C]$;

(iii) Using the expression from part (c) above, deduce $[L_A \circ L_B, L_C]$.