UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B3D: Pure Mathematics

COURSE CODE	:	**MATHB03D**
UNIT VALUE	:	**0.50**
DATE	:	**20–MAY–05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05–C0913–3–50

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcr} w & - & x & + & y & - & z & = & 2 \\ w & + & x & + & y & + & 3z & = & 4 \\ w & & & + & y & + & z & = & 3 \\ -w & + & x & - & y & + & z & = & -2 \end{array}$$

   (a) Write the system in the form of a matrix equation.

   (b) Reduce the augmented matrix of this system to echelon form. What is the rank of the augmented matrix?

   (c) Find the general solution of the system in the form $\underline{a} + s\underline{v}_1 + t\underline{v}_2$ where $\underline{a}$, $\underline{v}_1$ and $\underline{v}_2$ are vectors and s and t are scalars.

   (d) Apply the Gram-Schmidt process to the pair of vectors $\underline{v}_1$, $\underline{v}_2$ to give two new orthonormal vectors $\hat{\underline{g}}_1$ and $\hat{\underline{g}}_2$.

2. (a) Explain what is meant by eigenvalues and eigenvectors of a square matrix $\underline{\underline{A}}$.

   (b) Find the eigenvalues of the following matrix:

$$\underline{\underline{A}} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

   and find an eigenvector corresponding to each eigenvalue.

   (c) Hence find a matrix $\underline{\underline{V}}$ for which $\underline{\underline{V}}^{-1} = \underline{\underline{V}}^{\mathsf{T}}$ and $\underline{\underline{V}}^{\mathsf{T}}\underline{\underline{A}}\,\underline{\underline{V}}$ is diagonal. Write down the diagonal matrix that corresponds to your matrix $\underline{\underline{V}}$.

   (d) Write down the general solution $(x(t), y(t), z(t))^{\mathsf{T}}$ to the set of ordinary differential equations:

$$\begin{array}{rcrcrcr} \dot{x} & = & 3x & + & y & & \\ \dot{y} & = & x & + & 2y & + & z \\ \dot{z} & = & & & y & + & 3z \end{array}$$

   in which a dot indicates $\mathrm{d}/\mathrm{d}t$, i.e. $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and so on.

3. (a) Find the general solution to the differential equation

$$(t+2)\frac{\mathrm{d}f}{\mathrm{d}t} + 2f = \cos t.$$

(b) Find a particular solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 4x^3 + 2x^2 + x + 2.$$

Hence write down the general solution to this equation.

(c) A solution $y(x)$ to the equation in (b) above satisfies

$$y(0) = \frac{5}{4}; \qquad y'(0) = -\frac{5}{4}.$$

Find the explicit form of $y(x)$, and calculate $y(1)$.

4. The Fourier series for a function $f(x)$ with period T is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{T}\right)$$

(a) Write down formulae for the Fourier coefficients a_n and b_n.

(b) What can be said about the coefficients a_n and b_n if $f(x)$ is

(i) odd; (ii) even?

A function $f(x)$ with period $T = 4$ is defined by

$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 \le x \le 3 \\ 0 & 3 < x < 4 \end{cases}$$

(c) Sketch the graph of the function for three periods.

(d) Calculate the Fourier coefficients. Write down the first four non-zero terms in the Fourier series.

5. Let $f(x,y)$ be a real valued function of two variables x and y.

   (a) Explain what is meant by a critical point of the function $f(x,y)$.

   (b) Describe the possible types of *non-degenerate* critical point that can occur.

   (c) Give a condition in terms of second partial derivatives of f, under which the critical point is non-degenerate.

   Consider the function
$$f(x,y) = 3x^2 + 6xy + y^3 - 9y.$$

   (d) Find all of the first and second order partial derivatives of f.

   (e) Verify that $f_{xx}f_{yy} - f_{xy}^2 = 36(y-1)$.

   (f) Find and classify the two critical points of f.

6. Suppose you have a function $f(x,y)$ in which the variables x and y are themselves functions of s and t: i.e. $x = x(s,t)$ and $y = y(s,t)$, such that
$$F(s,t) = f(x(s,t), y(s,t)).$$

   (a) State the extended chain rule, which expresses the partial derivatives $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   Now suppose $x = s\cos t$ and $y = s\sin t$. Find expressions for:

   (b) $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$;

   (c) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$.

   (d) If a function $f(x,y)$ satisfies the two equations
$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0; \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x^2 + y^2,$$
   write down two partial differential equations in s and t which are satisfied by the corresponding function F.

   (e) Hence find the general form of $F(s,t)$ in this case.

7. (a) Explain what is meant by the *commutator* $[L_1, L_2]$ of two linear operators L_1 and L_2.

Find expressions for

(b) $\left[\frac{\mathrm{d}}{\mathrm{d}x}, x\frac{\mathrm{d}}{\mathrm{d}x}\right]$.

(c) $\left[\frac{\mathrm{d}}{\mathrm{d}x}, \left[e^x, \frac{\mathrm{d}}{\mathrm{d}x}\right]\right]$.

Now consider the three linear operators (acting on a two-dimensional vector)

$$L_A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad L_B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad L_C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the Pauli spin matrices from quantum mechanics.

(d) Show that $[L_B, L_C] = 2iL_A$.

(e) Find an expression for $[L_A, L_C]$.