UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *LL.B.* *M.Sci.*

Mathematics B3D: Pure Mathematics

COURSE CODE	:	**MATHB03D**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0909-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the following system of linear equations:

$$\begin{aligned} 6w - 6x + 12y &= 0 \quad , \\ 5w - 5x + 10y &= 0 \quad , \\ 3w - 3x + 6y + 2z &= 6 \quad . \end{aligned}$$

(a) Write the system in the form of a matrix equation.

(b) Use row operations to reduce the system to echelon form. What is the rank of the augmented matrix?

(c) Find the general solution of the system in the form $\mathbf{a} + \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$ where $\mathbf{a}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors and λ and μ are scalars.

(d) Find a vector $\mathbf{u}$ that is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is orthogonal to $\mathbf{v}_1$. Normalise $\mathbf{u}$ to have unit length.

2. For a function f of two variables x and y, describe the possible types of stationary (critical) points that can occur. Describe a test that usually would determine the type of a stationary point.

Consider the function

$$f(x, y) \;=\; (x^2 \;-\; y^2)\, e^x \quad .$$

By means of a sketch, indicate the regions of the (x,y) plane where f is positive and where f is negative.

Find and identify all the stationary points of f.

3. Explain what is meant by the eigenvectors and eigenvalues of a square matrix $\mathbf{A}$.

   (a) Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

   (b) Find a matrix $\mathbf{V}$ and a diagonal matrix Δ such that $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \Delta$. (You need not calculate $\mathbf{V}^{-1}$.)

   (c) Write the equation

$$3x^2 + 4xy + 3y^2 + 6z^2 = 1$$

   as a matrix equation. Explain how to transform this equation using normalised eigenvectors into an equation of the form

$$aX^2 + bY^2 + cZ^2 = 1$$

   and give values for the coefficients a, b and c. (You need not calculate X, Y, Z in terms of x, y, z.)

4. Suppose the variables x and y for a function $f(x, y)$ are themselves functions of s and t: i.e. $x = x(s, t)$ and $y = y(s, t)$. State the chain rule that gives expressions for the partial derivatives $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

   In what follows, f will denote the function $f(x, y) = (x^2 + y^2)\, x$.

   (a) Find the partial derivatives f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

   (b) Making the substitutions $x = r\cos(\theta)$, $y = r\sin(\theta)$, write $F(r, \theta) = f(x, y)$; use the chain rule to find F_r and F_θ in terms of r and θ.

   (c) Find $(rF_r)_r$ and $F_{\theta\theta}$, and hence find $r^{-1}(rF_r)_r + r^{-2}F_{\theta\theta}$. Compare this result to $f_{xx} + f_{yy}$.

5. The Fourier series for a function $f(x)$ with period T is

$$f(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} a_n \; cos(2n\pi x/T) + \sum_{n=1}^{\infty} b_n \; sin(2n\pi x/T)$$

(a) Write down formulae for the Fourier coefficients a_0, a_n, b_n.

A function $f(x)$ with period T=4 is defined by

$$f(x) \;=\; \begin{cases} 0 & -2 \;\leq\; x \;< -1 \\ x & -1 \;<\; x \;< 1 \\ 0 & 1 \;<\; x \;\leq 2 \end{cases}$$

(b) Sketch the shape of the function for three periods.

(c) Calculate the Fourier coefficients. Write down the first four non-zero terms in the Fourier series.

6. Suppose ϕ is a function of x and y in Cartesian co-ordinates. Give an expression for $\nabla\phi$ in terms of partial derivatives with respect to x and y.

Consider

$$\phi \;=\; 1 \;/\; (1 \;+\; 3x^2 \;+\; 4y^2)$$

(a) Calculate $\nabla\phi$.

(b) Find a unit vector $\mathbf{n}$ that points in the direction of most rapid increase of ϕ, (i) at $x = 0,\ y = 1$; (ii) at $x = 1,\ y = 1$; (iii) at $x = 1,\ y = -1$; (iv) at $x = -1,\ y = 0$.

(c) For the vector $\mathbf{v} \;=\; 4x\mathbf{i} \;-\; 3y\mathbf{j}$ calculate $\mathbf{v}\cdot\nabla\phi$.

(d) Along what lines does $\mathbf{v}\cdot\nabla\phi$ vanish?

7. Find particular solutions of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = f(t)$$

for (i) $f(t) = 10\sin(t)$, (ii) $f(t) = 9t^2 + 4$.
Find the general solution for $y(t)$ in case (i), and find an expression for $\frac{dy}{dt}$.
Hence find the solution in case (i) that satisfies the initial conditions

$$y(0) = 3 \quad , \quad y'(0) = 0 \ .$$

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