UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B3D: Pure Mathematics

COURSE CODE	:	**MATHB03D**
UNIT VALUE	:	**0.50**
DATE	:	**30-APR-03**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

03-C0901-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let V be the set of vectors $\mathbf{x}$ in $\mathbf{R}^5$ satisfying the following system of equations :

$$\begin{cases} x_1 \; -x_2 \qquad\quad -x_4 \; +x_5 & = \; 0 \\ x_1 \; -x_2 \; -x_3 & = \; 0 \\ x_1 \; -x_2 \; +x_3 \; -2x_4 \; +2x_5 & = \; 0 \end{cases} .$$

Find $\dim(V)$ and write down a basis for V.

By applying the Gram-Schmidt process, find an orthonormal basis for V.

2. The Fourier series of the real valued function $f(x)$ defined on $[-\pi, \pi]$ is

$$f(x) \;\sim\; C(f) + \sum_{m \geqslant 1} A_m(f)\cos(mx) + \sum_{m \geqslant 1} B_m(f)\sin(mx).$$

(i) Write down formulae for the Fourier coefficients $C(f)$, $A_m(f)$ and $B_m(f)$ as integrals involving f.

(ii) Show that $A_m(x^{2n}) = -\left(\frac{2n}{m}\right) B_m(x^{2n-1})$.

(iii) The Fourier series for the function $f(x) = x$ is

$$x \sim \sum_{m \geqslant 1} (-1)^{m+1} \frac{2}{m} \sin(mx);$$

Find the constant $C(x^2)$, and hence write down the Fourier series for the function

$$f(x) = x^2.$$

(iv) By integrating term by term, or otherwise, find the Fourier series of the function

$$f(x) = x^3 - \pi^2 x.$$

3. Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Hence find

(i) an orthogonal basis of eigenvectors of A ;

(ii) an invertible 3×3 matrix P such that $P^{-1}AP$ is diagonal.

If the matrix P is not orthogonal, explain briefly how to modify it until it becomes orthogonal.

4. Let $g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a real valued function of two variables x_1, x_2.

Explain what is meant by

(i) a critical point of g ;

(ii) a nondegenerate critical point.

Describe the possible types of behaviour of g near a nondegenerate critical point.

Let $$g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 6x_1^3 + x_2^2 + 6x_1x_2.$$

Show that g has a critical point at $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, and classify it according to type.

Show that g has precisely one other critical point. Find it, and classify it according to type.

5. Find a particular solution to the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 5\sin(x) - 15\cos(x).$$

Hence write down the general solution to this equation.

A solution $y(x)$ to the above equation satisfies

$$y(0) = 5 \quad ; \quad y'(0) = -5.$$

Find the explicit form of $y(x)$, and also the value of $y''(0)$.

6. By means of transformation to polar coordinates

$$x_1 = r\cos(\theta) \quad ; \quad x_2 = r\sin(\theta),$$

find expressions for the following differential operators in terms of r, θ.

$$(i)\ \frac{\partial}{\partial x_1} \ ; \ (ii)\ \frac{\partial}{\partial x_2} \ ; \ (iii)\ x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}.$$

A function $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfies the differential equation

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) f = (x_1^2 + x_2^2) \exp(\sqrt{x_1^2 + x_2^2}\,);$$

moreover, it is known that the value of f at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is independent of θ.

Find the explicit expression for $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in terms of x_1, x_2 if $f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1.$

7. Explain what is meant by the *commutator* $[\, L_1 \ , \ L_2 \,]$ of linear operators L_1, L_2. Find expressions for

$$(i)\ \left[\frac{d}{dx} \, , \, e^x \right] \quad ; \quad (ii)\ \left[\frac{d^2}{dx^2} \, , \, e^x \right].$$

The angular momentum operator in the (i, j)-plane is

$$L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}.$$

(iii) Show that $[\, L_{12} \ , \ L_{23} \,] = L_{13}$.

(iv) Find also an expression for

$$\left[x_1^3 \frac{\partial}{\partial x_2} \, , \, x_2^3 \frac{\partial}{\partial x_1} \right].$$

END OF PAPER