UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics B3D: Pure Mathematics

COURSE CODE	:	**MATHB03D**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0900-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **five** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Let V be the set of vectors $\mathbf{x} \in \mathbb{R}^5$ satisfying the following system of equations

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 + x_2 + x_4 + x_5 = 0 \end{cases}$$

Find $\dim(V)$ and write down a basis for V.
By applying the Gram-Schmidt process, find an orthonormal basis for V.

2. The Fourier series of a real valued function f defined on $[-\pi, \pi]$ is

$$f(x) = C(f) + \sum_{m\geq 1} A_m(f)\cos(mx) + \sum_{m\geq 1} B_m(f)\sin(mx).$$

i) Write down the formulae which express the Fourier coefficients $A_m(f)$, $B_m(f)$, $C(f)$ as integrals involving f.

ii) Show that

$$B_m(x^{2n+1}) = (-1)^{m+1}\frac{2\pi^{2n}}{m} + \left(\frac{2n+1}{m}\right) A_m(x^{2n}).$$

iii) What can one say about $A_m(x^{2n+1})$, $C(x^{2n+1})$?

iv) The Fourier series of $f(x) = x^2$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{m\geq 1}(-1)^m \frac{4}{m^2}\cos(mx).$$

Find the Fourier series of $g(x) = x^3$.

3. Find an orthogonal basis for $\mathbb{R}^3$ consisting of eigenvectors of the following matrix

$$A = \begin{pmatrix} 3 & 5 & 0 \\ 5 & 3 & 12 \\ 0 & 12 & 3 \end{pmatrix}$$

Hence find an invertible matrix P such that $P^{-1}AP$ is diagonal.
If your matrix P is not orthogonal, show how to modify it until it becomes orthogonal.

4. If $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a real valued function of two real variables x_1, x_2, explain what is meant by saying that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ is a nondegenerate critical point of } f.$$

List the possible types of nondegenerate critical points, together with numerical criteria which allow one to distinguish between them.

$$\text{Let } f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \exp(x_1^2 + x_1x_2 + x_2^2 + 3x_1).$$

Show that f has precisely one critical point. Find it. Show it is nondegenerate and find its type.

5. Explain what is meant by the commutator $[L_1, L_2]$ of linear operators L_1, L_2. Find expressions for

i) $\left[\frac{d}{dx},\ x^2\right]$;

ii) $\left[\frac{d^2}{dx^2},\ \exp(x)\right]$;

iii) $\left[x_1^3\frac{\partial}{\partial x_2},\ x_2^3\frac{\partial}{\partial x_1}\right]$;

iv) $\left[x_1\frac{\partial}{\partial x_2},\ x_2\frac{\partial}{\partial x_3}\right]$.

6. Find a particular solution to the differential equation

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 10y = 10x^3 - 9x^2 + 4x - 3.$$

Hence write down the general solution.

A solution $y(x)$ to the above equation satisfies

$$y(0) = 7\ ;\ y'(0) = -1.$$

Write down the explicit form of $y(x)$, and calculate

i) $y(1)$ and

ii) $y''(0)$.

7. By means of transformation to polar coordinates

$$x_1 = r\cos(\theta)\ ;\ x_2 = r\sin(\theta)$$

find expressions for the following differential operators:

$$\text{i)}\ \frac{\partial}{\partial x_1}\ ;\quad \text{ii)}\ \frac{\partial}{\partial x_2}\ ;\quad \text{iii)}\ x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}.$$

A function $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ takes the form

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r^2 h(\theta)$$

and satisfies

$$\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2.$$

Find $h(\theta)$ if $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi$