UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics C393: Probability

COURSE CODE	:	**MATHC393**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1024-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. State and prove the second Borel-Cantelli Lemma.

   A fair coin is tossed repeatedly. Let k be a natural number. Prove that, almost surely, there will occur infinitely many sequences of k consecutive heads. Deduce that almost surely, there will be arbitrarily long sequences of heads.

   Let H_n be the number of heads after n tosses, T_n the number of tails and let S_n be the difference, $H_n - T_n$. Show that S_n is almost surely unbounded.

2. Prove that if two π-systems are independent then so are their generated σ-algebras. (You may assume that if two probability measures defined on a σ-algebra agree on a π-system generating the σ-algebra, then they agree.)

   Let $(\mathcal{F}_n)$ be an independent sequence of σ-algebras. Show that for any n and $m > n$, the σ-algebras $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \mathcal{F}_n)$ and $\sigma(\mathcal{F}_{n+1} \cup \mathcal{F}_{n+2} \cup \ldots \mathcal{F}_m)$ are independent.

   State and prove Kolmogorov's $0-1$ Law.

3. Let X be a random variable with mean 0 and variance 1, and let ϕ be the characteristic function of X. Show that ϕ is a C^2 function and that $\phi''(0) = -1$.

   Let (X_i) be an IID sequence of random variables, each with the law of X. For each natural number n, find the characteristic function ϕ_n of

$$\frac{1}{\sqrt{n}} \sum_1^n X_i$$

   in terms of the characteristic function of X.

   Prove that

$$\phi_n(t) \rightarrow e^{-\frac{1}{2}t^2}$$

   uniformly on bounded intervals of $\mathbf{R}$.

   In a few lines, explain why this fact is interesting.

4. Let $(\mathcal{F}_n)$ be a filtration on a probability space Ω. What does it mean to say that T is a *stopping time* with respect to the filtration.

For such a T, let $\mathcal{F}_T$ be the collection of sets A with the property that for each n,

$$A \cap (T = n) \in \mathcal{F}_n.$$

Show that $\mathcal{F}_T$ is a σ-algebra.

For a martingale (X_n) with respect to the filtration, define the random variable X_T and show that it is $\mathcal{F}_T$-measurable.

Suppose that $T \leqslant m$ everywhere. Show that

$$\mathrm{E}(X_m | \mathcal{F}_T) = X_T.$$

5. State Kolmogorov's Strong Law of Large Numbers and deduce that if (X_i) is an IID sequence of integrable random variables with $\mathrm{E}X_i = \theta < 0$ (for each i), then

$$\sum_1^n X_i \to -\infty \quad \text{a.s.}$$

Let U be a positive random variable satisfying

$$\begin{aligned} \mathrm{E}U &= 1 \quad \text{and} \\ \mathrm{E}|\log U| &< \infty. \end{aligned}$$

Show that $\mathrm{E}\log U \leqslant 0$ and that $\mathrm{E}\log U < 0$ unless $U = 1$ a.s.

Now suppose that (U_i) is an IID sequence of random variables with the distribution of U and for each n let $X_n = \prod_1^n U_i$. Show that either

$$X_n = 1 \text{ for all } n, \text{ a.s.}$$

or

$$X_n \to 0 \text{ a.s.}$$