UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics C393: Probability

COURSE CODE	:	**MATHC393**
UNIT VALUE	:	**0.50**
DATE	:	**17-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the two Borel-Cantelli Lemmas.
   (b) Assume $A_1, A_2, \ldots$ are events in a probability space $(\Omega, \Sigma, \mathbb{P})$. Does $\mathbb{P}(A_n \text{ i.o.}) = 0$ imply $\sum \mathbb{P}(A_n) < \infty$?
   (c) Let $X_1, X_2, \ldots$ be IID RVs with exponential distribution of parameter 1. Set
   $$X = \limsup \frac{X_n}{\log n}.$$
   Show that $X = 1$ almost surely.
   (d) Let $X_1, X_2, \ldots$ be random variables such that
   $$\mathbb{P}(X_n = -1) = 1 - 1/n^2 \text{ and } \mathbb{P}(X_n = n^2 - 1) = 1/n^2$$
   and set $S_n = \frac{1}{n}(X_1 + \ldots + X_n)$. Prove that $\mathbb{E}X_n = 0$ for every n but $\mathbb{P}(\lim S_n = -1) = 1$. [Hint: Show that, with probability one, all but finitely many X_n are equal to -1].

2. (a) State and prove Chebyshev's inequality.
   (b) Prove that if X is a random variable in $\mathcal{L}^r$ and $1 \leqslant p \leqslant r$, then X is in $\mathcal{L}^p$ and
   $$(\mathbb{E}|X|^p)^{\frac{1}{p}} \leqslant (\mathbb{E}|X|^r)^{\frac{1}{r}}.$$
   (c) Suppose that X, Y are independent random variables on $(\Omega, \Sigma, \mathbb{P})$ and $X, Y \in \mathcal{L}^1$. Show that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.
   (d) Let $X_1, \ldots, X_n$ be IID RVs with (the same) continuous distribution function. Let B_k denote the event that $X_k > X_i$ for all $i = 1, 2, \ldots, k-1$ (i.e., X_k is a record). Show that $P(B_n) = 1/n$. Are the events B_k, B_n $(k < n)$ independent?

3. (a) Prove the strong law of large numbers for IID RVs $X_1, X_2, \ldots$ assuming that $\mathbb{E}|X_n|^4 < \infty$.
   (b) State and prove the Bernstein-Hoeffding inequality.
   (c) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of σ-algebras. Define the tail σ-algebra. State and prove Kolmogorov's $0-1$ law.

4. (a) Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, X an integrable random variable, and $\mathcal{F}$ a sub-σ-algebra of Σ. Define the conditional expectation $\mathbb{E}(X|\mathcal{F})$. Prove that it exists, and that if f and g are two versions of $\mathbb{E}(X|\mathcal{F})$ then $f = g$ almost surely.

   (b) Let X be a random variable with mean 0 and variance 1. Define the characteristic function ϕ of X and prove that ϕ is in C^2, with $\phi'(0) = 0$ and $\phi''(0) = -1$.

   (c) Let $X_1, X_2, \ldots$ be a sequence of IID RVs with common characteristic function ϕ. Find the characteristic function of

$$\frac{1}{\sqrt{n}} \sum_{1}^{n} X_i$$

   in terms of ϕ.

5. Write an essay on martingales.