UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C393: Probability

COURSE CODE : **MATHC393**

UNIT VALUE : **0.50**

DATE : **20-MAY-04**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

04-C0936-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the two Borel-Cantelli Lemmas.

   (b) Suppose $(A_n)_{n=1}^{\infty}$ are events such that $\mathbb{P}(A_n, \text{ i.o.}) = 0$. Must it be the case that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$?

   (c) Suppose $(A_n)_{n=1}^{\infty}$ are events such that $\mathbb{P}(A_n) \to 0$ as $n \to \infty$. Must we have $\mathbb{P}(A_n, \text{ i.o.}) = 0$?

   (d) Suppose $X_1, X_2, \ldots$ are independent random variables such that X_n is uniformly distributed on $[0, n]$ for every n. Prove that $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty) = 0$.

   (e) Suppose $X_1, Y_1, X_2, Y_2, \ldots$ are independent random variables such that X_n and Y_n are uniformly distributed on $[0, n]$ for every n. Prove that $\mathbb{P}(X_n + Y_n \to \infty \text{ as } n \to \infty) = 1$.

2. (a) State what it means for two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ to be independent. What does it mean for two random variables X and Y to be independent?

   (b) Prove that if X, Y are independent random variables in $\mathcal{L}^1$ then $XY \in \mathcal{L}^1$ and $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

   (c) Suppose that X and Y are random variables such that $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$. Must X and Y be independent?

   (d) Prove that if $X_1, \ldots, X_n$ are independent then $\text{var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{var}(X_i)$.

   (e) Suppose $X_1, \ldots, X_n$ are independent random variables with $\mathbb{E}X_i = 0$ for every i and $\sum_{i=1}^{n} X_i = 0$. Prove that $\mathbb{P}(X_i = 0) = 1$ for every i.

3. (a) Prove that if T_1 and T_2 are independent π-systems then $\sigma(T_1)$ and $\sigma(T_2)$ are independent. [You may assume that if two probability measures agree on a π-system $\mathcal{C}$ then they agree on $\sigma(\mathcal{C})$.]

   (b) Prove that if $X_1, X_2, \ldots$ are independent random variables and $n > 1$ then $\sigma(X_1, \ldots, X_n)$ and $\sigma(X_{n+1}, X_{n+2}, \ldots)$ are independent.

   (c) Suppose that $X_1, X_2, \ldots$ are random variables and $\sigma(X_1, \ldots, X_n)$ is independent from $\sigma(X_{n+1})$ for every n. Must $X_1, X_2, \ldots$ be independent?

   (d) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be σ-algebras. Define the *tail σ-algebra*. State and prove Kolmogorov's 0-1 Law.

4. (a) Let X be a random variable with mean 0 and variance 1. Define the characteristic function ϕ of X and prove that ϕ is C^2, with $\phi'(0) = 0$ and $\phi''(0) = -1$.

(b) Let $(X_n)_{n=1}^{\infty}$ be an IID sequence of random variables with the same distribution as X, and define Y_n by

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i.$$

Find the characteristic function ϕ_n of Y_n in terms of the characteristic function ϕ of X.

(c) Prove that $\phi_n(t) \to e^{-t^2/2}$ uniformly on bounded intervals of $\mathbb{R}$. [You may assume that the principal value of the logarithm satisfies $|\mathrm{Log}(1+z) - z| \leq |z|^2$ for $|z| < 1/2$.]

5. Write an essay on martingales.