UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. M.Sci.

Mathematics C393: Probability

COURSE CODE	:	**MATHC393**
UNIT VALUE	:	**0.50**
DATE	:	**22-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0930-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the two Borel-Cantelli Lemmas.

   (b) Prove that if $K > 0$ is a real number, and $X_1, X_2, \ldots$ are random variables such that $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^4 < K$ for every i, then

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \to 0 \text{ as } n \to \infty\right) = 1.$$

   (c) Let $X_1, X_2, \ldots$ be random variables such that

$$X_n = \begin{cases} n^2 - 1 & \text{with probability } 1/n^2 \\ -1 & \text{with probability } 1 - 1/n^2 \end{cases}$$

   for every n. Prove that $\mathbb{E}X_n = 0$ for every n, but

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \to -1 \text{ as } n \to \infty\right) = 1.$$

   [Hint: Show that, with probability 1, all but finitely many X_n are equal to -1.]

2. (a) Define the term *π-system.*

   (b) Prove that if two probability measures agree on a π-system then they agree on the σ-algebra it generates.

   (c) Let P_1 and P_2 be two probability measures on the measurable space (Ω, Σ). Let $\mathcal{F} = \{X \in \Sigma : P_1(X) = P_2(X)\}$. Must P_1 and P_2 agree on $\sigma(\mathcal{F})$?

3. (a) State what it means for two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ to be independent. What does it mean for two random variables X and Y to be independent?

(b) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of σ-algebras. Define a *tail event.* State and prove Kolmogorov's 0-1 Law.

[You may assume that if T_1 and T_2 are independent π-systems then $\sigma(T_1)$ and $\sigma(T_2)$ are independent.]

(c) Let $X_1, X_2, \ldots$ be an independent sequence of random variables, and let $S_n = \sum_{i=1}^{n} X_i$. Suppose that S_n/n converges almost surely as $n \to \infty$. Prove that there is some number a such that $S_n/n \to a$ almost surely.

[Hint: Let $L = \lim_{n\to\infty} S_n/n$. For each real number a, consider the event $(L \geqslant a)$.]

4. (a) Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, X an integrable random variable and $\mathcal{F} \subset \Sigma$ a σ-algebra. Define what it means for a function f to be a version of the conditional expectation $\mathbb{E}(X|\mathcal{F})$. Prove that the conditional expectation exists, and that if f and g are versions of the conditional expectation then $f = g$ almost surely.

(b) Define the terms *filtration, submartingale, martingale* and *stopping time.*

(c) State and prove the Optional Stopping Theorem.

(d) Prove that if $(X_k)_{k=1}^{\infty}$ is a nonnegative submartingale then

$$\mathbb{E}X_n \geqslant t\mathbb{P}(\max_{0\leqslant k\leqslant n} X_k \geqslant t).$$

5. Write an essay on the Central Limit Theorem.