UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics C393: Probability

COURSE CODE	:	**MATHC393**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0925-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the two Borel-Cantelli Lemmas.

   (b) Suppose that $(X_n)_{n=1}^{\infty}$ is a sequence of random variables such that, for every $\epsilon > 0$,
$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \epsilon) < \infty.$$
   Prove that $X_n \to 0$ almost surely.

   (c) State and prove Chebyshev's Inequality.

   (d) Let $(X_n)_{n=1}^{\infty}$ be independent random variables with mean 0 and variance 1. Prove that if $\alpha > 1/2$ then
$$\frac{X_n}{n^\alpha} \to 0 \text{ almost surely.}$$

2. (a) Let $(\mathcal{F}_n)_{n=1}^{\infty}$ be an independent sequence of σ-algebras. Show that $\sigma(\mathcal{F}_1, \dots, \mathcal{F}_n)$ and $\sigma(\mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \dots)$ are independent. [You may assume that if two π-systems are independent then so are the σ-algebras they generate.]

   (b) State and prove Kolmogorov's 0-1 Law.

   (c) Define what it means for a sequence $(X_n)_{n=1}^{\infty}$ of random variables to be independent.

   (d) Prove that if $(X_n)_{n=1}^{\infty}$ is an independent sequence of random variables then either $(X_n)_{n=1}^{\infty}$ is bounded almost surely or $(X_n)_{n=1}^{\infty}$ is unbounded almost surely.

3. (a) State Hölder's Inequality for real-valued functions f and g on a probability space $(\Omega, \Sigma, \mathbb{P})$. Deduce that if f and g are in $\mathcal{L}^2$ then fg is in $\mathcal{L}^1$.

   (b) Prove that if X is a random variable in $\mathcal{L}^r$ and $1 \leq p < r$ then X is in $\mathcal{L}^p$ and
$$(\mathbb{E}|X|^p)^{1/p} \leq (\mathbb{E}|X|^r)^{1/r}.$$

   (c) Prove that if X and Y are *independent* random variables in $\mathcal{L}^1$ then $XY \in \mathcal{L}^1$ and $\mathbb{E}(XY) = \mathbb{E}X \cdot \mathbb{E}Y$.

   (d) Give an example of random variables X and Y in $\mathcal{L}^1$ such that $XY \notin \mathcal{L}^1$.

4. Let $(\mu_n)_{n=1}^{\infty}$, μ be probability measures on $\mathbb{R}$ with distributions $(F_n)_{n=1}^{\infty}$, F and characteristic functions $(\phi_n)_{n=1}^{\infty}$, ϕ.

(a) Define what it means to say that $\mu_n \to \mu$ weakly.

(b) Prove that if

$$\int h d\mu_n \to \int h d\mu \text{ as } n \to \infty \tag{1}$$

for every bounded continuous function $h : \mathbb{R} \to \mathbb{R}$, then μ_n tends weakly to μ.

(c) Prove that if $\phi_n(t) \to \phi(t)$ uniformly on bounded intervals then $\mu_n \to \mu$ weakly. [You may assume Parseval's theorem, and that it is sufficient to prove (1) for C^2 functions with compact support.]

5. Write an essay on martingales.

END OF PAPER