UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C315: Numerical Analysis I

COURSE CODE	:	**MATHC315**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1005-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Suppose $y(t)$ is continuously differentiable at least three times in the domain $[t_0, t_N]$. Write down the first four terms of the Taylor series for $y(t+h)$ about the point t in the domain, where h is some small constant stepsize.

   Consider the initial value problem

$$dy/dt = f(y) \quad , \quad y(t_0) = y_0 \quad ,$$

   where the solution is required in $[t_0, t_N]$, and $f(y)$ is continuously differentiable at least twice.

   Show that $d^3y/dt^3 = ff_yf_y + f^2 f_{yy}$, where f_y=df/dy etc. .

   An algorithm to estimate y numerically is given by $y_{n+1} = y_n + hG(y_n; h)$, where

$$G(y_n;h) = [\, f(y_n) + 4f(\, y_n + \frac{1}{2}hf(y_n)\,) + f(\, y_n - hf(y_n) + 2hf(\, y_n + \frac{1}{2}hf(y_n)\,)\,)\,]\,/\,6\,,$$

   and y_n is the estimate of y at $t_n = t_0 + nh$.

   (a) What type of method is this algorithm?

   (b) Show that to $O(h^2)$

$$f(\, y(t_n) + \frac{1}{2}hf_n\,) = f_n + \frac{1}{2}hf_nf_{yn} + \frac{1}{8}h^2 f_nf_nf_{yyn}\,,$$

   where f_n here denotes $f(y(t_n))$, f_{yn} denotes the derivative of f at $y(t_n)$, etc.

   (c) The truncation error of the method is defined as

$$T_n = (\, y(t_{n+1}) - y(t_n)\,)/h - G(y(t_n); h) \quad .$$

   Prove that T_n is $O(h^3)$, by showing that T_n is zero to $O(h^2)$.

   (d) For the particular case $f(y) = \lambda y$, where λ is constant, show that the algorithm becomes

$$y_{n+1} = y_n + (\mu/6)(6 + 3\mu + \mu^2)y_n \quad ,$$

   where $\mu = h\lambda$.

2. Consider the initial value problem

$$dy/dt = f(y,t) \quad , \quad y(t_0) = y_0 \quad ,$$

to be solved in the domain $[t_0, t_N]$, with f continuously differentiable in the domain of interest. Suppose y_n is the estimate of y at $t_n = t_0 + nh$, with constant stepsize $h = (t_N - t_0)/N$.

Describe in general how an Adams-Bashforth explicit multistep scheme of order $p+1$ makes use of estimates of f at t_n, t_{n-1}, ... t_{n-p} to estimate y_{n+1}.

For the case p=1, derive the formula

$$y_{n+1} = y_n + (h/2)(3f_n - f_{n-1}) \quad , \qquad (1)$$

where f_n denotes $f(y_n, t_n)$.

The algorithm requires 2 starting values: how might those starting values be obtained?

Consider the case where f=$ay + b$, where a and b are non-zero constants, and the initial condition is $y(0)$=$1 - b/a$.

(a) What is the exact solution $y(t)$ to the initial value problem in this case?

(b) Write (1) as a linear difference equation, and find the general solution for y_n, which will will contain a particular solution and two linearly independent homogeneous solutions with arbitrary coefficients. (You may use the notation α=$ah/2$ to simplify expressions.)

(c) Use the initial condition to find a relation between the arbitrary coefficients.

(d) For $|ah| \ll 1$, find approximate expressions for $y(t_1)$ and y_1 to order ah. By equating these, obtain another relation between the arbitrary coefficients and hence determine their value.

3. Suppose $f(x)$ is a continuous real function with a zero at $x=r$, x_n is the estimate of r at step n of an iterative zero-finding algorithm, and $e_n = x_n - r$ is the error at step n. Define the order of convergence of such an algorithm.

Describe the secant method, using a sketch to illustrate your answer.

(a) What are the advantages of this method compared to (i) the bisection, and (ii) the Newton-Raphson methods?

(b) When the errors e_n are small, show that approximately

$$e_{n+1} = \gamma e_n e_{n-1} \quad ,$$

where $\gamma = f''(r)/2f'(r)$, assuming the first two derivatives of f at r exist and are non-zero.

(c) Hence show that the order of convergence of the method is $p=(1+\sqrt{5})/2$.

(d) Suppose the secant method is used to evaluate $\ln(a)$ by calculating a zero of the function $f(x) = e^x - a$, where a is a positive constant. If the error after n iterations is ϵ, make use of the relation in (b) to show that the error after two further iterations is approximately $(\epsilon/2)^p\ \epsilon$.

4. (a) Given that a polynomial of degree N can have at most N distinct zeroes, prove that the polynomial $P(x)$ that fits $N+1$ points (x_n, y_n) must be unique.

(b) Given that a function $f(x)$ is continuously differentiable three times in the interval $[a,b]$, and that $P(x)$ is a quadratic polynomial that is equal to $f(x)$ at the points x_0, x_1 and x_2 in $[a,b]$, prove that for each x in $[a,b]$ there is a point η in $[a,b]$ such that

$$f(x) - P(x) = f'''(\eta)(x-x_0)(x-x_1)(x-x_2)/6 \quad .$$

(c) Suppose $x_j = x_0 + jh$ for some constant stepsize h. Prove that the maximum absolute value of $(x-x_0)(x-x_1)(x-x_2)$ in the interval $[x_0, x_2]$ is $2(h/\sqrt{3})^3$.

(d) Quadratic polynomials can be used to approximate a function $f(x)$ by fitting the values of f at x_0, x_1 and x_2 by one quadratic, fitting values at x_2, x_3 and x_4 by another quadratic, etc. . Estimate the number of equally-spaced points needed to approximate $f(x) = xe^{-x}$ in this way with absolute error less than 10^{-3} throughout the range $[0,2]$.
(You may use that approximately $\sqrt{3} = 1.732$, $1/\sqrt{3} = 0.5777$.)

5. Describe the cubic spline interpolation function $S(x)$ that fits $N+1$ data points (x_n, y_n) for $n = 0, 1, ..., N$.

   (a) How many coefficients need to be determined, and what are the constraints that can be used to find the coefficients?

   (b) What do the terms 'natural' and 'clamped' cubic spline mean?

   (c) Calculate the natural cubic spline that fits the data points (0,4), (1,-2) and (2,0). (Hint: use the constraints directly, rather than a matrix method.)

   Suppose **L** is an MxM matrix with constant elements. The elements of the main diagonal are all equal to 1, and the only other non-zero elements are in the diagonal immediately below the main diagonal. In general, how many multiplication operations are needed to calculate the M elements of the vector **y**, when **Ly**=**b**, and **b** is a vector with given values?

   Describe briefly why this result is relevant to the efficiency of calculating a cubic spline to fit a large number of points.

END OF PAPER