UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics C315: Numerical Analysis I

COURSE CODE	:	**MATHC315**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0920-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Suppose a function $y(t)$ has at least $m+1$ continuous derivatives in the interval [a,b], and t_0 is a point in [a,b]. Using Taylor's theorem, write $y(t)$ as the sum of a polynomial of degree m in t-t_0 and a remainder term.

   Consider the initial value problem

   $$dy/dt = f(y) \quad , \quad y(t_0) = y_0 \quad .$$

   A one-step numerical scheme to estimate the solution at points $t_n = t_0 + nh$ can be written in the form

   $$y_{n+1} = y_n + hG(y_n, t_n; h) \quad ,$$

   where y_n is the estimate of $y(t_n)$, and h is the constant step length.

   (a) Write down the second order Taylor scheme (Taylor2) in terms of $f(y)$ and its derivative, and a general form for the second order Runge-Kutta method (RK2). Why is the Taylor scheme seldom used in practice?
   By comparing these two schemes, or otherwise, find constraints on the parameters in the RK2 scheme, and deduce that there is a one parameter family of RK2 schemes.

   (b) The truncation error is defined as

   $$T_n = (y(t_{n+1}) - y(t_n))/h - G(y(t_n), t_n; h) \quad .$$

   Find expressions for T_n to $O(h^2)$ in terms of $f(y)$ and its derivatives for
   (i) Taylor2, and
   (ii) RK2 .

   (c) For the particular case $f(y) = y^2$, find expressions for these two versions of T_n as functions of y. Which method is likely to be more accurate in this case?

2. The forward difference operator Δ is defined as $\Delta y_n = y_{n+1} - y_n$.
Write down an expression for $\Delta^2 y_n$.

(a) Consider the difference equation

$$y_{n+2} - y_{n+1} - 2y_n = -2 \quad .$$

(i) Find the general solution for y_n.
(ii) Find the solution when the initial conditions are $y_0 = 0$ and $\Delta y_0 = 2$. Describe what might happen if the solution is calculated numerically using just the initial conditions and the difference equation.

(b) Consider the initial value problem

$$dy/dt = -y/a \quad , \quad y(0) = 1 \quad , \quad \text{where } a \text{ is a positive constant.}$$

With step length h, Euler's method for the numerical solution is

$$\Delta y_n = -hy_n/a \quad .$$

Find the solution of this difference equation, with $y_0 = 1$. Describe the behaviour for various ranges of values of h: e.g. $h << a$, $h = a$, etc.
Compare this solution with the exact solution of the differential equation.

3. Consider the ordinary differential equation

$$dy/dt = f(y,t) \quad .$$

Two possible algorithms for estimating the solution at the points $t_n = t_0 + nh$ are

$$(A) \qquad y_{n+1} = y_n + hf_n \qquad ,$$

and

$$(B) \qquad y_{n+1} = y_n + (h/2)(f_n + f_{n+1}) \quad ,$$

where $f_n = f(y_n, t_n)$.

(a) What types of methods are these (i.e. explicit, single-step, ...)? Discuss their likely advantages and disadvantages.

(b) Define the growth factor for the error ϵ_n that arises when calculating solutions numerically using such algorithms. By considering the special case $f = \lambda y$, where λ is a real constant, analyse the stability of schemes (A) and (B), giving a range of values of λh for which each scheme is stable.

(c) The above methods can be combined in a predictor-corrector scheme of the form

$$y^*_{n+1} = y_n + hf_n \quad ;$$

$$y_{n+1} = y_n + (h/2)\,(f_n + f^*_{n+1}) \quad ;$$

where $f^*_{n+1} = f(y^*_{n+1}, t_{n+1})$.

With $f = \lambda y$ as before, analyse the stability of this new scheme.

4. Consider a function $f(x)$ which has a real root at $x = r$. Suppose an algorithm provides successive estimates $x_0, x_1, ..., x_n, ...$ for the root r. In terms of the error $\epsilon_n = x_n - r$, define what is meant by the order of convergence of the algorithm.

(a) Describe the Newton-Raphson (N-R) method for finding a root of $f(x)$, which uses information about the derivative df/dx . Illustrate your description with a sketch. What difficulties can arise with this method?

(b) For $f(x) = a + x^2 e^x$, analyse the convergence of the N-R method
(i) when $a = -4$ (you may assume the root is positive in this case), and
(ii) when $a = 0$. (Hint: simplify the algorithm first by putting in the functional forms for f and df/dx .)
Why is the order of convergence different in these two cases?

(c) For the case $a = 0$, substitute the term $f(x_n)$ in the N-R method by $\gamma f(x_n)$, where γ is a constant. Find a value for γ that would increase the order of convergence of the modified scheme.

5. Suppose data values (x_n, y_n) are given for $n = 0, ..., N$.

(a) Discuss (without details) the main advantages and disadvantages of using a single polynomial $P_N(x)$ to interpolate such data.

(b) Find coefficients such that $P_3(x)$ fits the four data points (-1,4), (0,1), (1,0), and (2,-5).

(c) Suppose a function $y(x)$ is at least M+1 times continuously differentiable in a domain $[a, b]$. If $P_M(x)$ is the interpolating polynomial that fits y at points x_k in the domain for k=0,...,M, then it can be shown that

$$\epsilon(x) = y(x) - P_M(x) = y^{(M+1)}(\eta) \prod_{k=0}^{M}(x - x_k) \; / \; (M + 1)!$$

for some point η in the domain.
Use this property to obtain a formula that bounds $|\epsilon|$ for linear interpolation between two points x_0 and x_1 separated by a distance h.

(d) Consider the case $y(x) = e^{-x^2/2}$. Suppose you are asked to approximate y by using linear interpolation with N+1 equally-spaced points in the domain [-1,1]. Estimate a value of N such that the interpolation error is less than 10^{-6} throughout this domain.